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Fractal models of defects growth in materials

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Introduction

Machines working and constructions stability depend on features of materials used to build their elements. In fatigue process these features are changed.

Natural question is if changes of some material features may be chaotic?

The main goal of the work is a description of fractal defects evolution in material with methods of dynamical systems theory.

The multiplicative calculus

Additive calculus		Multiplicative calculus
$a + b$	$\rightleftarrows$	$a * b$
$a * b$		$a^{\ln b} = b^{\ln a}$

Multiplicative derivative of function $f(x)$ with respect to x:

$$\frac{\pi f(x)}{\pi x} = \lim_{\epsilon \to 0} \left\{ \frac{f(x(1+\epsilon))}{f(x)} \right\}^{\frac{1}{\epsilon}}$$

V. Volterra, B. Hostinsky *Operations Infinitesimales Lineares*. Herman, Paris, 1938.

Physical stability of physical systems

Evolution of physically unstable systems may be chaotic.

In physics, superheating means the heating of a liquid to a temperature above its normal boiling point. It take place when heating is too long, container has a very smooth surface and water is very clean, then bubbles are not formed.

Superheating is metastable, the water boils vigorously with small interfering.

Animation from:
http://www.phys.unsw.edu.au/ jw/superheating.html

Fractal models of defects growth

based on single fractal approximation

Assuming uniform energy distribution over fractal describing defects in mesoscopic and macroscopic range:

$$\mathcal{E} = a(D, \delta)\nu_D$$

where:

D is Hausdorff fractal dimension,
ν_D is Hausdorff fractal measure,

δ is Hausdorff metric describing how defects fill volume V of material sample, for sufficiently small δ Minkowski estimation is obeyed:

$$\nu_D \delta^{3-D} = V = \mathrm{const}$$

$a(D, \delta)$ is density of energy accumulated on defects.

Fractal models of defects growth

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Energy shift $\epsilon_{\mathcal{E}}$ calculated from multiplicative first order expansion of function $\mathcal{E}(D, \nu_D)$:

$$\epsilon_{\mathcal{E}} = (1-\kappa)D\epsilon_{\nu} + (1-\kappa)\epsilon_{D} \ln\left(\frac{\nu_D}{\mathbb{A}(D)}\right)$$

where $\epsilon_{\mathcal{E}}$, ϵ_D, ϵ_ν are values shifts:

$$\mathcal{E}' = (1+\epsilon_{\mathcal{E}})\mathcal{E},\ D' = (1+\epsilon_D)D,\ \nu_D' = (1+D\epsilon_\nu)\nu_D,$$

characteristic measure $\mathbb{A}(D)$ is linear combination of characteristic measure $\mathcal{A}(D)$ i external field: limited volume V of material sample:

$$\mathbb{A}(D) = \mathcal{A}(D)^{\frac{1}{1-\kappa}}(\delta^D)^{\frac{-\kappa}{1-\kappa}}, \quad \mathcal{A}(D) = \left(\frac{\pi a}{\pi D}\right)^{-1}, \quad \kappa = \frac{\ln\left(\frac{\pi a}{\pi \delta}\right)}{3-D}$$

Fractal models of defects growth

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Numeric analysis of physical stability proved that model is not physically stable.

The characteristic measure $\mathcal{A}(D)$ has singularities, critical effect appears and is connected with jump of $\mathcal{A}(D)$ from ∞ to 0.

Defects growth is similar to phase transition.

In Ginzburg - Landau theory of continuous phase transitions the thermodynamic potentials are expanded into power series according to order parameter, which is intensive quantity of system.

Order parameter ϕ has value > 0 in asymmetrical, ordered phase, and equals 0 in symmetrical, disordered phase. Phase transition from more to less symmetric phase is symmetry breaking.

Fractal models of defects growth

based on single fractal approximation

In fractal model of defects growth the characteristic measure $\mathcal{A}(D)$ plays the role of the order parameter.

When $\mathcal{A}(D)$ is ∞ or 0, scaling symmetry is occuring, defects change in the same way independently from dimension.

Scaling symmetry can be broken under external field, which appears with limited volume of material sample.

Scaling symmetry is broken before and after critical effect. Before critical effect defects change regularly and the fractal measure increases proportionally, Evolution of autonomous fragments of defects is not correlated. After critical effect it is correlated and whole defect grows homogeneously.

Fractal models of defects growth

based on single fractal approximation

Analogue of the Ginzburg-Landau theory of continuous phase transitions has been employed to create new model of fractal defects growth with higher order expansion of energy:

$$\mathcal{E} = a(D,\delta)\nu_D + b(D,\delta)\nu_D{}^2$$

where:

$a(D,\delta)$ is density of energy accumulated on defects,

$b(D,\delta)$ concerns internal interaction of fractal approximating defects and describes nonuniform energy distribution over single fractal defect.

Fractal models of defects growth

based on single fractal approximation

Multiplicative expansion of energy equals:

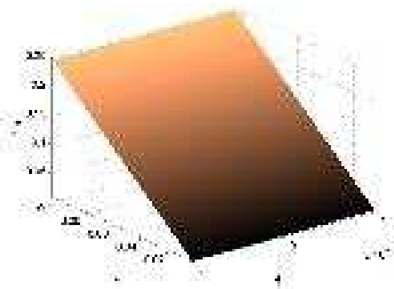
$$\epsilon_E = \frac{1 - \kappa + 2\frac{\nu_D}{\alpha}}{1 + \frac{\nu_D}{\alpha}} \left(D\epsilon_\nu + \epsilon_D \ln\left(\frac{\nu_D}{\mathbb{A}(D)}\right)\right)$$

$$\mathbb{A}(D) = (\mathcal{A}(D))^{\frac{1}{1-\kappa+2\frac{\nu_D}{\alpha}}} (\mathcal{B}(D))^{\frac{2\frac{\nu_D}{\alpha}}{1-\kappa+2\frac{\nu_D}{\alpha}}} \left(\delta^D\right)^{\frac{-\kappa}{1-\kappa+2\frac{\nu_D}{\alpha}}}$$

$$\frac{\pi a}{\pi D} = (\mathcal{A}(D))^{-1}, \quad \frac{\pi b}{\pi D} = (\mathcal{B}(D))^{-2}, \quad \frac{b(D,\delta)}{a(D,\delta)} = \frac{1}{\alpha}$$

$$\kappa = \kappa_a + \frac{\nu_D}{\alpha}\kappa_b, \quad \kappa_a = \frac{\ln\frac{\pi a}{\pi\delta}}{3-D}, \quad \kappa_b = \frac{\ln\frac{\pi b}{\pi\delta}}{3-D}$$

Model is physically stable for multiplicative expansion of derivatives $\frac{\pi a}{\pi\delta}$ i $\frac{\pi b}{\pi\delta}$.



Results of numerical analysis of model physical stability.

Fractal models of defects growth

based on two fractals approximation

Models of fractals defects growth based on dimension decomposition of measures:

- Independent fractals:

$$\mathcal{E} = a_1(D_1)\nu_{D_1} + a_2(D_2)\nu_{D_2}$$

- Fractals depending on their dimensions:

$$\mathcal{E} = a_1(D_1, D_2)\nu_{D_1} + a_2(D_2, D_1)\nu_{D_2}$$

- Fractals depending on their measures:

$$\mathcal{E} = a_1(D_1, \nu_{D_2})\nu_{D_1} + a_2(D_2, \nu_{D_1})\nu_{D_2}$$

Fractal models of defects growth

based on two fractals approximation

Fractal model of defects growth:

$$\mathcal{E} = a(D_1, D_2)\nu_{D_1} + a(D_1, D_2)\nu_{D_2} + b(D_1, D_2)\nu_{D_1}\nu_{D_2}$$

- is composed of fractals growth energies and the energy of their interaction,
- assume symmetry of energy density of fractal growth on the base of Mean Field Approximation,
- $b(D_1, D_2)$ describes nonuniform energy distribution over fractal defects.

Fractal models of defects growth

based on two fractals approximation

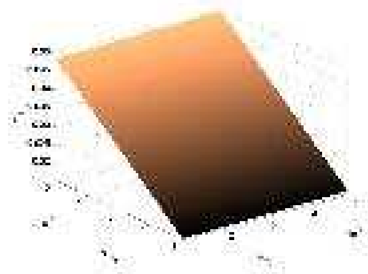
Multiplicative expansion of energy equals:

$$\epsilon_E = \frac{\nu_{D_1} + \alpha\nu_{D_1}\nu_{D_2}}{\nu_{D_1} + \nu_{D_2} + \alpha\nu_{D_1}\nu_{D_2}} \left(D_1 \epsilon_{\nu_1} + \epsilon_{D_1} \ln \frac{\nu_{D_1}}{A_1(D_1, D_2, \nu_{D_1}, \nu_{D_2})} \right) + \frac{\nu_{D_2} + \alpha\nu_{D_1}\nu_{D_2}}{\nu_{D_1} + \nu_{D_2} + \alpha\nu_{D_1}\nu_{D_2}} \left(D_2 \epsilon_{\nu_2} + \epsilon_{D_2} \ln \frac{\nu_{D_2}}{A_2(D_1, D_2, \nu_{D_1}, \nu_{D_2})} \right)$$

where:

$$\alpha = \frac{b(D_1, D_2)}{a(D_1, D_2)}$$

Model is physically stable.



Results od numerical analysis of physical stability of model.

Evolution of characteristic measures

Energy shift ϵ_ε depends on two characteristic measures: $\mathbb{A}_1$ i $\mathbb{A}_2$:

$$\mathbb{A}_1(D_1, D_2, \nu_{D_1}, \nu_{D_2}) = \mathcal{A}(D_1)^{\frac{\nu_{D_1}+\nu_{D_2}}{\nu_{D_1}+\alpha\nu_{D_1}\nu_{D_2}}} \mathcal{B}(D_1)^{\frac{\alpha\nu_{D_1}\nu_{D_2}}{\nu_{D_1}+\alpha\nu_{D_1}\nu_{D_2}}}$$

$$\mathbb{A}_2(D_1, D_2, \nu_{D_1}, \nu_{D_2}) = \mathcal{A}(D_2)^{\frac{\nu_{D_1}+\nu_{D_2}}{\nu_{D_2}+\alpha\nu_{D_1}\nu_{D_2}}} \mathcal{B}(D_2)^{\frac{\alpha\nu_{D_1}\nu_{D_2}}{\nu_{D_2}+\alpha\nu_{D_1}\nu_{D_2}}}$$

where:

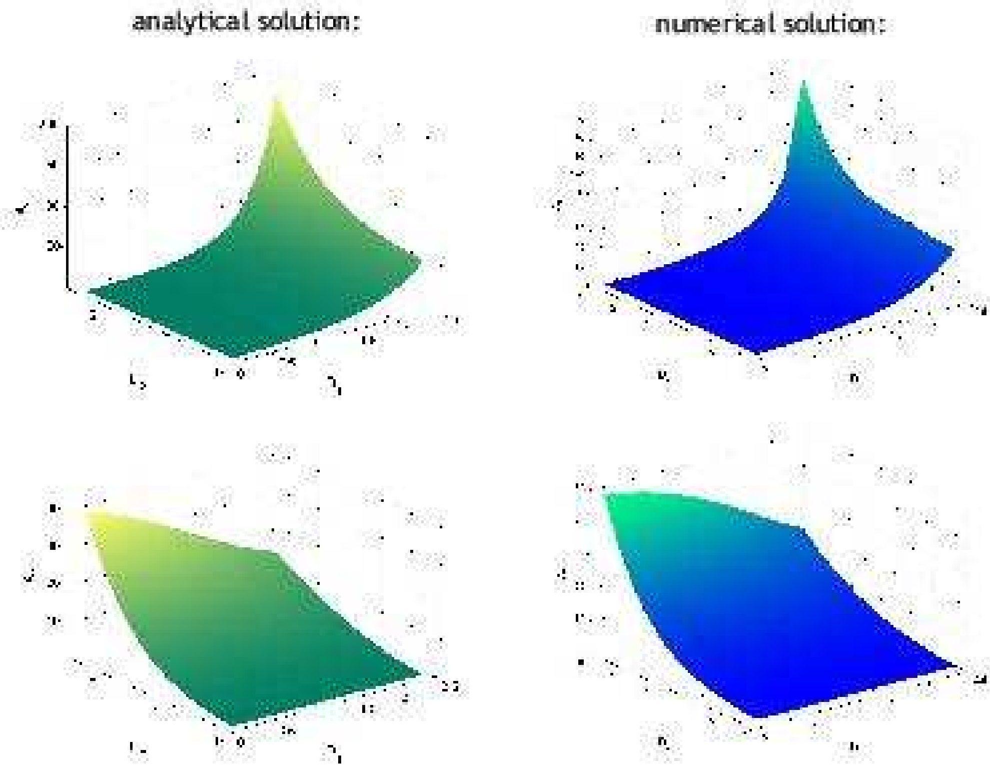
$$\mathcal{A}(D_i) = \left(\frac{\pi a}{\pi D_i}\right)^{-1}, \quad \mathcal{B}(D_i) = \left(\frac{\pi b}{\pi D_i}\right)^{-1},$$

Multiplicative partial derivatives of $\mathbb{A}_1$ and $\mathbb{A}_2$ give dynamical system:

$$\frac{\pi \mathbb{A}_1}{\pi D_1}, \frac{\pi \mathbb{A}_1}{\pi D_2}, \frac{\pi \mathbb{A}_1}{\pi \nu_{D_1}}, \frac{\pi \mathbb{A}_1}{\pi \nu_{D_2}}, \frac{\pi \mathbb{A}_2}{\pi D_1}, \frac{\pi \mathbb{A}_2}{\pi D_2}, \frac{\pi \mathbb{A}_2}{\pi \nu_{D_1}}, \frac{\pi \mathbb{A}_2}{\pi \nu_{D_2}}$$

Evolution of characteristic measures

Results of numerical simulations



Example of evolution of characteristic measures A_1 and A_2.

Disscusion

- Physically stable fractal model of defects growth in material based on single fractal approximation containing higher order expansion of energy
- Confirmation of critical effect occuring in physically stable fractal model of defects growth
- Physically stable fractal model of defects growth based on two fractal approximation with fractal interaction
- Application of dynamical systems theory to describe evolution of characteristic measures of fractal defects

Thank you for your attention