6.3 Maxwell–Mohr Method

The Maxwell–Mohr procedure presents a universal method for computation of displacement at any point of any deformable structure. Also, the Maxwell–Mohr procedure allows calculating mutual displacements. Different sources, which may cause displacements of a structure, are considered. They are different types of loads and change of temperature.

6.3.1 Deflections Due to Fixed Loads

For bending systems, the Castigliano's theorem for computation of linear and angular displacements at point k may be presented as follows

$$y_k = \int \frac{M(x)}{EI}\frac{\partial M(x)}{\partial P_k}\mathrm{d}x, \quad \theta_k = \int \frac{M(x)}{EI}\frac{\partial M(x)}{\partial P_k}\mathrm{d}x, \tag{6.8}$$

where $M(x)$ is bending moment at section x, P_k and M_k are force and couple at section k.

Both formulas (6.8) may be simplified. For this purpose let us consider, for example, the simply supported beam subjected to force P and couple M (Fig. 6.9).

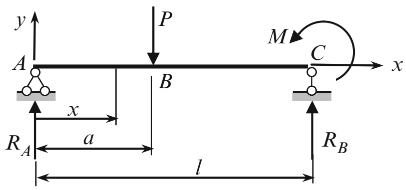

Fig. 6.9 Simply supported beam loaded by P and M

Reaction

$$R_A = P\frac{l-a}{l} + M\frac{1}{l}$$

and the bending moment for the left and right portions of the beam are

$$M(x) = R_A x = P\frac{(l-a)}{l}x + M\frac{x}{l} \quad (x \le a),$$

$$M(x) = R_A x - P(x-a) = P\frac{a}{l}(l-x) + M\frac{x}{l} \quad (x \ge a).$$

Both expressions present the *linear* functions of the loads P and M. In general case, suppose a structure is subjected to the set of concentrated loads P_1, $P_2, \ldots$, couples $M_1, M_2, \ldots$, and distributed loads $q_1, q_2, \ldots$. This condition of structure is called as P-condition (also known as the actual or loaded condition). In case of P–condition, a bending moment at the any section x is a *linear function* of these loads

$$M(x) = a_1 P_1 + a_2 P_2 + \cdots + b_1 M_1 + b_2 M_2 + \cdots + c_1 q_1 + c_2 q_2 + \cdots, \quad (6.9)$$

where coefficients a_i, b_i, and c_i depend on geometrical parameters of the structure, position of loads, and location of the section x.

If it is required to find displacement at the point of application of P_1, then, as an intermediate step of Castigliano's theorem we need to calculate the partial derivative of bending moment $M(x)$ with respect to force P_1. This derivative is $\partial M(x)/\partial P_1 = a_1$. According to expression for $M(x)$, this parameter a_1 may be considered as the bending moment at section x caused by *unit dimensionless force* ($P_1 = 1$). State of the structure due to action of unit dimensionless load (unit force or unit couple) is called *unit state*. Thus, calculation of partial derivatives in (6.8) may be changed by calculation of a bending moment caused by unit dimensionless load

$$y_k = \int \frac{M(x)}{EI} \frac{\partial M(x)}{\partial P} \mathrm{d}x = \int \frac{M(x)\bar{M}_k}{EI} \mathrm{d}x, \quad (6.10)$$

where $\bar{M}_k$ is bending moment in the unit state. Keep in mind that $\bar{M}_k$ is always a *linear function* and represents the bending moment due to a unit load, which corresponds to the required displacement.

In a similar way, terms, which take into account influence of normal and shear forces, may be transformed. Thus, displacements caused by any combination of loads may be expressed in terms of internal stresses developed by given loads and *unit* load, which corresponds to required displacement. That is the reason why this approach is termed the dummy load method. A general expression for displacement may be written as

$$\Delta_{kp} = \sum \int_0^s \frac{M_p \bar{M}_k}{EI} \mathrm{d}s + \sum \int_0^s \frac{N_p \bar{N}_k}{EA} \mathrm{d}s + \sum \int_0^s \mu \frac{Q_p \bar{Q}_k}{GA} \mathrm{d}s. \quad (6.11)$$

Summation is related to all elements of a structure. Fundamental expression (6.11) is known as Maxwell–Mohr integral. The following notations are used: Δ_{kp} is displacement of a structure in the kth direction in P-condition, i.e., displacement in the direction of unit load (first index k) due to the given load (second index p); M_p, N_p, and Q_p are the internal stresses (bending moment, axial and shear forces) in

P-condition; and $\bar{M}_k$, $\bar{N}_k$, $\bar{Q}_k$ are the internal stresses due to the unit load, which acts in the kth direction and corresponds to the required displacement. GA is transversal rigidity, μ is non-dimensional parameter depends on the shape of the cross-section. For rectangular cross section this parameter equals 1.2, for circular section it equals 10/9. The unit load (force, couple, etc.) is also termed as the dummy load.

For different types of structures, relative contribution of first, second, and third terms of expression (6.11) in the total displacement Δ_{kp} is different. For practical calculation, depending on type and shape of a structure, the following terms from (6.11) should be taken into account:

(a) For trusses – only second term
(b) For beams and frames with ratio of height of cross section to span 0.2 or less – only first term
(c) For beams with ratio of height of cross section to span more than 0.2 – the first and third terms
(d) For gently sloping arches – the first and second terms
(e) For arches with ratio of radius of curvature to height of cross section 5 or more – all terms

In case of trusses, the displacement should be calculated by formula

$$\Delta_{kp} = \sum \int_0^l \frac{N_p \bar{N}_k}{EA} \mathrm{d}s. \tag{6.12}$$

Since all elements are straight ones and axial stiffness EA is constant along all length of each element, then this formula may be presented as

$$\Delta_{kp} = \sum \frac{N_p \bar{N}_k}{EA} l. \tag{6.13}$$

Procedure for computation of deflections using Maxwell–Mohr integral is as follows:

1. Express internal forces in P-condition for an arbitrary cross section in terms of its position x
2. Construct the unit condition. For this we should apply unit load (dummy load), which corresponds to the required displacement:
   (a) For linear displacement, a corresponding dummy load represents the unit force, which is applied at the point where displacement is to be determined and acts in the same direction
   (b) For angular displacement, a corresponding dummy load is the unit couple, which is applied at the point where angle of rotation is to be determined

(c) For mutual linear displacement of two sections, a corresponding dummy load represents two unit forces, which are applied at the points where displacement is to be determined and act in the opposite directions
(d) For mutual angular displacement of two sections, a corresponding dummy load represents two unit couples, which are applied at given sections and act in the opposite directions.

3. Express the internal forces in unit condition for an arbitrary cross section in terms of its position x
4. Calculate Maxwell–Mohr integral

Positive sign of displacement means that the real displacement coincides with the direction of the unit load, or work performed by unit load along the actual direction is positive.

Example 6.7. A cantilever uniform beam is subjected to a uniformly distributed load q (Fig. 6.10a). Compute (a) the angle of rotation and (b) vertical displacement at point A. Take into account only bending moments.

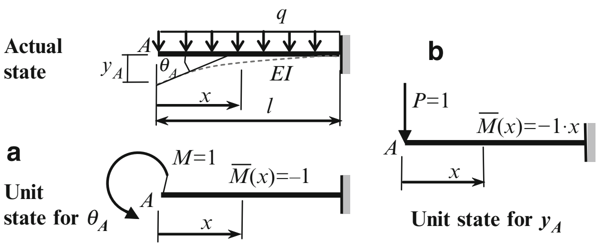

Fig. 6.10 Design diagram of the beam; (**a**) Unit state for θ_A; (**b**) Unit state for y_A

Solution. (a) The angle of rotation may be defined by formula

$$\theta_A = \frac{1}{EI}\int_0^l M_p(x)\bar{M}\,\mathrm{d}x. \tag{a}$$

Now we need to consider two states, mainly, the actual and unit ones, and for both of them set up the expressions for bending moments. For actual state, the bending moment is $M_p(x) = -qx^2/2$. Since it is required to determine the slope at point A, then the unit state presents the same structure with unit couple $M = 1$ at point A (Fig. 6.10a); this dummy load may be shown in arbitrary direction. For unit state, the bending moment is $\bar{M} = -1$ for any section x. The formula (a) for required angle of rotation becomes

$$\theta_A = \frac{1}{EI}\int_0^l \left(-\frac{qx^2}{2}\right)\cdot(-1)\mathrm{d}x = \frac{ql^3}{6EI}. \tag{b}$$

(b) The vertical displacement at A may be calculated by formula

$$y_A = \frac{1}{EI}\int_0^l M_p(x)\bar{M}\,\mathrm{d}x, \tag{c}$$

where expression for bending moment $M_p(x)$ in the actual state remains without change. In order to construct the unit state, it is necessary to apply unit concentrated force $P = 1$ at the point where it is required to determine displacement (Fig. 6.10b).

For unit state, the bending moment is $\bar{M} = -1 \cdot x$. The formula (c) for vertical displacement becomes

$$y_A = \frac{1}{EI}\int_0^l \left(-\frac{qx^2}{2}\right)\cdot(-1\cdot x)\mathrm{d}x = \frac{ql^4}{8EI}. \tag{d}$$

The positive sign means that adopted unit load make positive work on the real displacement, or other words, actual displacement coincides with assumed one.

Example 6.8. Determine the vertical displacement of joint 6 of symmetrical truss shown in Fig. 6.11. Axial rigidity for diagonal and vertical elements is EA and for lower and top chords is $2EA$.

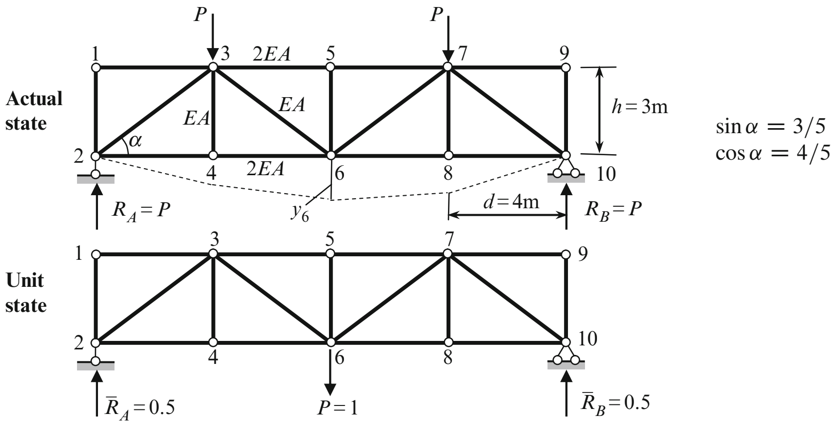

Fig. 6.11 Design diagram of the truss (actual state) and unit state

Solution. All elements of the given structure are subjected to axial loads only, so for required displacement the following formula should be applied:

$$y_6 = \sum \frac{1}{EA} N_p \bar{N}_k l, \tag{a}$$

where N_p and $\overline{N}_k$ are internal forces in actual and unit state, respectively.