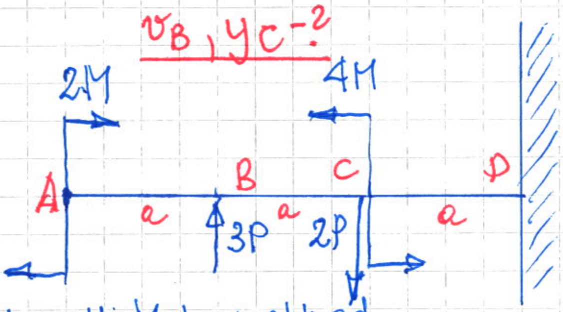
A

Ex. 1. $P, a, M = P \cdot a$, $EI = const$

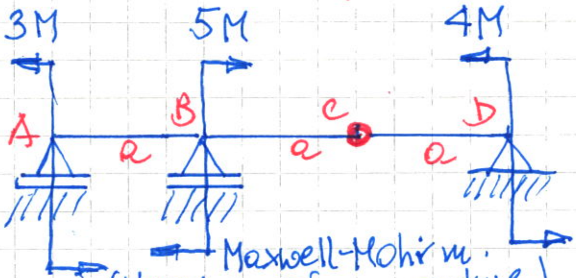
ϑ_B, y_C - ?



Maxwell-Mohr method
(Vereshchagin procedure)
(or graph multiplication method)

Ex. 3. $M, a, EI = const$

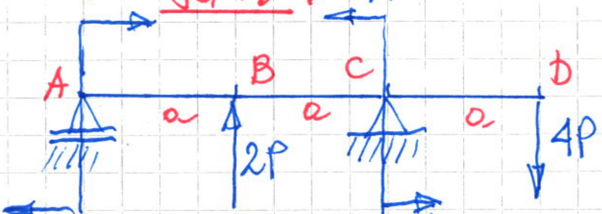
ϑ_B, ϑ_D - ?



Maxwell-Mohr m.
(Vereshchagin procedure)
or graph multiplication method

Ex. 2. $P, a, M = P \cdot a$, $EI = const$

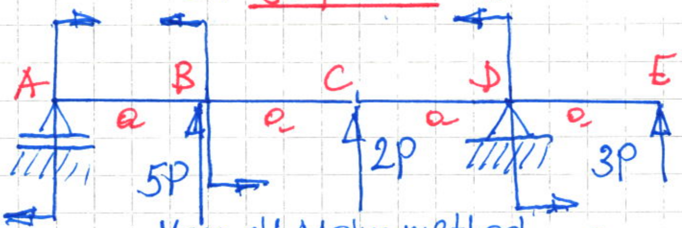
y_B, ϑ_D - ?



Maxwell-Mohr method
(Vereshchagin procedure)
(or graph multiplication method)

Ex. 4. $P, a, M = P \cdot a$, $EI = const$

y_B, ϑ_E - ?

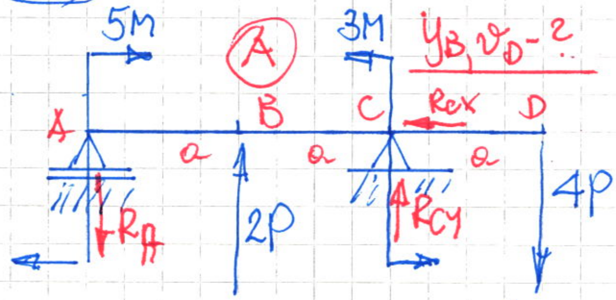


Maxwell-Mohr method
(Vereshchagin procedure)
(or graph multiplication method)

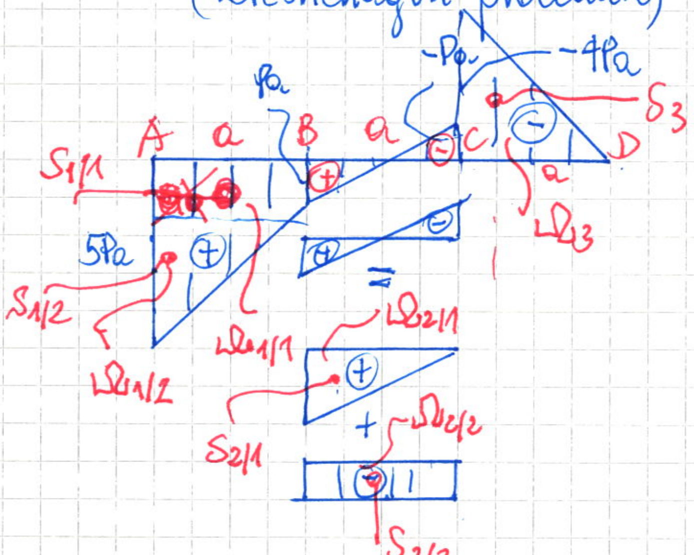
Ex. 2. $P, a, M = Pa, EI = \text{const}$

y_B, ϑ_D - ?

A



Maxwell - Mohr method (Vereshchagin procedure)



$\underline{2a \le x_3 \le 3a}$

$M_3(x) = 3Pa - 2P \cdot x - 3M + 6P(x - 2a) =$
$3Pa - 2Px - 3Pa + 6P \cdot x - 12Pa = 4P \cdot x - 12Pa$

I Static eqs.

① $\sum P_{ix} = -R_{Cx} = 0$

② $\sum P_{iy} = -R_A + 2P + R_{Cy} - 4P = 0$

③ $\sum M_i^C = 5M - R_A \cdot 2a + 2P \cdot a - 3M + 4P \cdot a = 0$

$5Pa - R_A \cdot 2a + 2Pa - 3Pa + 4Pa = 0$

$-R_A \cdot 2a + 8Pa = 0$

$R_A = 4P$

② $\Rightarrow -4P + 2P + R_{Cy} - 4P = 0$

$R_{Cy} = 6P$

II Bending moments

$\underline{0 \le x_1 \le a}$

$M_1(x) = 5M - R_A \cdot x = 5Pa - 4P \cdot x$
$= P(5a - 4x)$

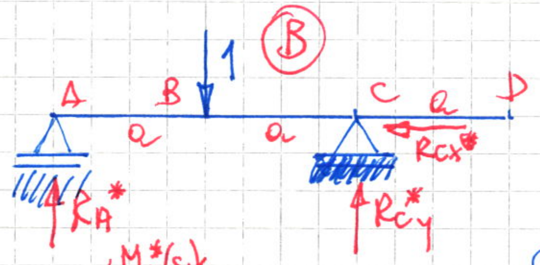
$M_1(0) = 5Pa, \quad M_1(a) = Pa$

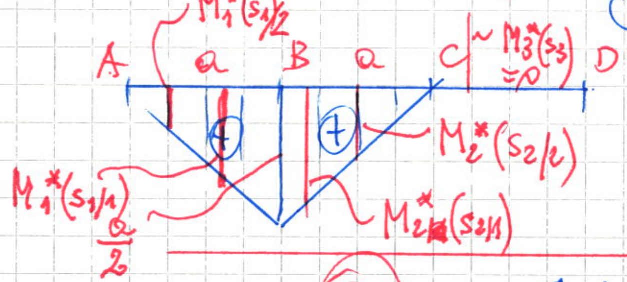
$\underline{a \le x_2 \le 2a}$

$M_2(x) = 5M - R_A \cdot x + 2P(x - a) =$
$= 5Pa - 4Px + 2Px - 2Pa = 3Pa - 2Px$

$M_2(a) = Pa, \quad M_2(2a) = 3Pa - 4Pa = -Pa$

B





I Static eqs.

① $\sum P_{ix} = -R_{Cx}^* = 0$

② $R_A^* + R_{Cy}^* - 1 = 0 = \sum P_{iy}$

③ $\sum M_i^C = R_A^* \cdot 2a - 1 \cdot a = 0 \qquad R_A^* = \frac{1}{2}$

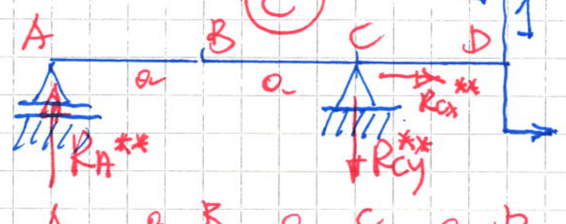
② $\Rightarrow \frac{1}{2} + R_{Cy}^* - 1 = 0 \qquad R_{Cy}^* = \frac{1}{2}$

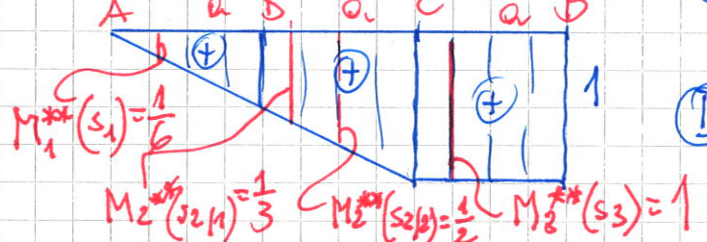
II Bending moments

$M_1^*(x) = R_A^* \cdot x = \frac{1}{2} \cdot x = \frac{x}{2}$

$M_1^*(a) = \frac{a}{2}$

C





I Static eqs.

① $\sum P_{ix} = R_{Cx}^{**} = 0$

② $\sum P_{iy} = R_A^{**} - R_{Cy}^{**} = 0 \qquad R_A^{**} = R_{Cy}^{**}$

③ $\sum M_i^C = R_A^{**} \cdot 2a - 1 = 0$

$R_A^{**} = R_{Cy}^{**} = \frac{1}{2a}$

II Bending moments

$M_{1/2}^{**}(x) = R_A^{**} \cdot x = \frac{1}{2a} \cdot x$

$$y_B = \frac{1}{EI}\left[\underbrace{\Omega_{1/1}}_{A}\cdot \underbrace{M_1^*(s_{1/1})}_{B} + \underbrace{\Omega_{1/2}}_{A}\cdot \underbrace{M_1^*(s_{1/2})}_{B} + \underbrace{\Omega_{2/1}}_{A}\cdot \underbrace{M_2^*(s_{2/1})}_{B} + \underbrace{\Omega_{2/2}}_{A}\cdot \underbrace{M_2^*(s_{2/2})}_{B} + \right.$$

$$\left. + \underbrace{\Omega_3}_{A} \cdot \underbrace{\overset{=0}{M_3^*(s_3)}}_{B}\right] =$$

$\Omega_{1/1} = Pa^2$, $\Omega_{1/2} = 2Pa^2$, $\Omega_{2/1} = Pa^2$, $\Omega_{2/2} = -Pa^2$, $\Omega_{3} = -2Pa^2$

$M_1^*(s_{1/1}) = \frac{a}{4}$, $M_1^*(s_{1/2}) = \frac{a}{6}$, $M_2^*(s_{2/1}) = \frac{a}{3}$, $M_2^*(s_{2/2}) = \frac{a}{4}$

$M_3^*(s_3) = 0$

$$y_B = \frac{1}{EI}\left[Pa^2\cdot\frac{a}{4} + 2Pa^2\cdot\frac{a}{6} + Pa^2\cdot\frac{a}{3} - Pa^2\cdot\frac{a}{4} - 2Pa^2\cdot 0\right] =$$

$$= \frac{1}{EI}Pa^3\left(\frac{1}{4}+\frac{1}{3}+\frac{1}{3}-\frac{1}{4}-2\cdot 0\right) = \frac{2}{3}\frac{Pa^3}{EI}$$

$$y_B = \frac{2}{3}\frac{Pa^3}{EI}$$

M-M analytical

$$y_B = \frac{1}{EI}\left[\int_0^a M_1(x)\cdot M_1^*(x)\,dx + \int_a^{2a} M_2(x)\cdot M_2^*(x)\,dx + \int_{2a}^{3a} M_3(x)\cdot M_3^*(x)\,dx\right] = \ldots$$

$M_1(x) = 5Pa - 4P\cdot x$, $M_2(x) = 3Pa - 2P\cdot x$, $M_3(x) = 4P\cdot x - 12Pa$

$M_1^*(x) = \frac{x}{2}$, $M_2^*(x) = a - \frac{x}{2}$, $M_3^*(x) = 0$

$$y_B = \frac{1}{EI}\left[\int_0^a (5Pa - 4P\cdot x)\cdot\frac{x}{2}\,dx + \int_a^{2a}(3Pa - 2P\cdot x)\left(a - \frac{x}{2}\right) + \int_{2a}^{3a}(4Px - 12\cdot Pa)\cdot 0\,dx\right] =$$

$$\ldots = \frac{2}{3}\frac{Pa^3}{EI}$$

$$\vartheta_D = \frac{1}{EI}\Big[\underbrace{\Omega_{1/1}}_{A}\cdot \underbrace{M_1^*(s_{1/1})}_{C} + \underbrace{\Omega_{1/2}}_{A}\cdot \underbrace{M_1^*(s_{1/2})}_{C} + \underbrace{\Omega_{2/1}}_{A}\cdot \underbrace{M_2^*(s_{2/1})}_{C} + \underbrace{\Omega_{2/2}}_{A}\cdot \underbrace{M_2^*(s_{2/2})}_{C}$$

$$+ \underbrace{\Omega_3}_{A}\cdot \underbrace{M_3^*(s_3)}_{C}\Big] = ?$$

$\Omega_{1/1} = Pa^2$, $\Omega_{1/2} = 2Pa^2$, $\Omega_{2/1} = Pa^2$, $\Omega_{2/2} = -Pa^2$, $\Omega_3 = -2Pa^2$

$M_1^*(s_{1/1}) = \frac{1}{4}$, $M_1^*(s_{1/2}) = \frac{1}{6}$, $M_2^*(s_{2/1}) = \frac{2}{3}$ $M_2^*(s_{2/2}) = \frac{3}{4}$, $M_3^*(s_3) = 1$

$$\vartheta_D = \frac{1}{EI}\left[Pa^2\cdot\frac{1}{4} + 2Pa^2\cdot\frac{1}{6} + Pa^2\cdot\frac{2}{3} - Pa^2\cdot\frac{3}{4} - 2Pa^2\cdot 1\right] =$$

$$= \frac{1}{EI}Pa^2\left[\frac{1}{4} + \frac{1}{3} + \frac{2}{3} - \frac{3}{4} - 2\right] = -\frac{3}{2}\frac{Pa^2}{EI}$$

M-M analytical

$$\vartheta_D = \frac{1}{EI}\left[\int_0^a M_1(x)\cdot M_1^*(x)\,dx + \int_a^{2a} M_2(x)\cdot M_2^*(x)\,dx + \int_{2a}^{3a} M_3(x)\cdot M_3^*(x)\,dx\right] = ?$$

$M_1(x) = 5Pa - 4P\cdot x$, $M_2(x) = 3Pa - 2P\cdot x$, $M_3(x) = 4P\cdot x - 12Pa$

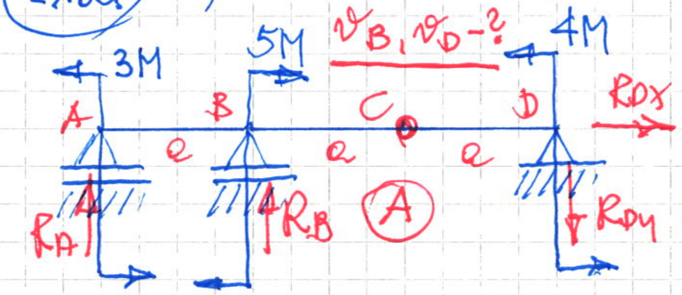
$M_1^*(x) = \frac{x}{2a}$, $M_2^*(x) = \frac{x}{2a}$, $M_3^*(x) = 1$

$$\vartheta_D = \frac{1}{EI}\left[\int_0^a (5a - 4P\cdot x)\cdot\frac{x}{2a}\,dx + \int_a^{2a}(3Pa - 2P\cdot x)\cdot\frac{x}{2a}\,dx + \int_{2a}^{3a}(4P\cdot x - 12Pa)\cdot 1\,dx\right] =$$

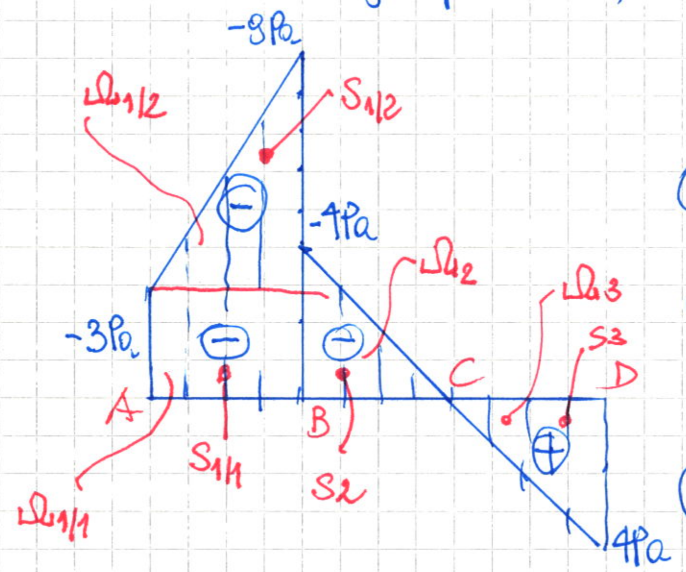
$$= \ldots -\frac{3}{2}\frac{Pa^2}{EI}$$

Ex. 3 P, M, a, EI = const, M = P·a

ϑ_B, ϑ_D - ?



Maxwell-Mohr method
(Vereshchagin procedure)



I static eqs

(1) $\sum P_{ix} = R_{Dx} = 0$

(2) $\sum P_{iy} = R_A + R_B - R_{Dy} = 0$

(3) $M_C^L = R_A \cdot 2a - 3M + 5M + R_B \cdot a = 0$

$R_A \cdot 2a - 3Pa + 5Pa + R_B \cdot a = 0$

$R_A \cdot 2a + 2Pa + R_B \cdot a = 0 \quad | : a$

$2R_A + 2P + R_B = 0$

(4) $M_C^R = -4M + R_{Dy} \cdot a = 0$

$-4Pa + R_{Dy} \cdot a = 0 \qquad R_{Dy} = 4P$

(2) $\Rightarrow R_A + R_B = R_{Dy} = 4P$

$R_B = 4P - R_A$

(3) $\Rightarrow 2R_A + 2P + R_B = 2\cancel{R_A} + 2P + 4P - \cancel{R_A} = 0$

$R_A + 6P = 0 \Rightarrow R_A = -6P$

$R_B = 4P - R_A = 4P + 6P = 10P = R_B$

II Bending moments

$0 \le x_1 \le a$

$M_1(x) = R_A \cdot x - 3M = -6P \cdot x - 3Pa = -3P(2x+a)$

$M_1(x) = -3P(2x+a)$, $M_1(0) = -3Pa$, $M(a) = -9Pa$

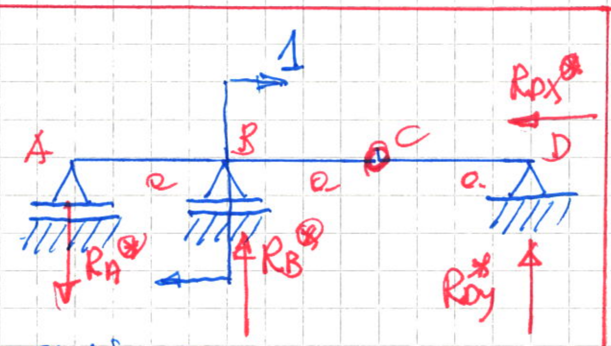
$a \le x_2 \le 2a$

$M_2(x) = R_A \cdot x - 3M + 5M + R_B(x-a) =$
$= -6P \cdot x - 3Pa + 5Pa + 10P(x-a) =$
$= -6P \cdot x + 2Pa + 10Px - 10Pa = 4Px - 8Pa$

$M_2(a) = -4Pa$, $M_2(2a) = 0$

$2a \le x_3 \le 3a$

$M_3(x) = R_A \cdot x - 3M + 5M + R_B(x-a) =$
$4P \cdot x - 8Pa$; $M_3(3a) = 4Pa$



I static eqs.

(1) $\sum P_{ix} = -R_{Dx}^* = 0$

(2) $\sum P_{iy} = -R_A^* + R_B^* + R_{Dy}^* = 0$

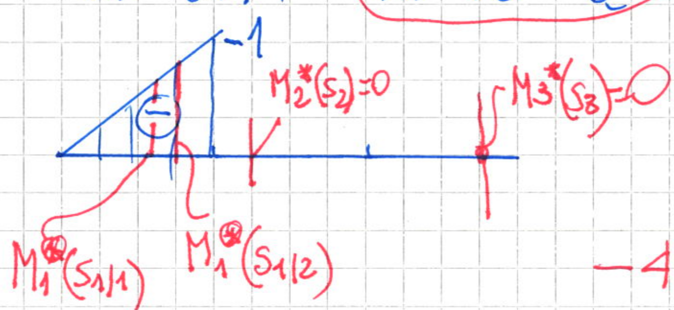
(3) $M_C^L = -R_A^* \cdot 2a + 1 + R_B^* \cdot a = 0$

(4) $M_C^R = -R_{Dy}^* \cdot a = 0 \Rightarrow R_{Dy}^* = 0$

(2) $\Rightarrow -R_A^* + R_B^* = 0 \Rightarrow R_A^* = R_B^*$

(3) $\Rightarrow -R_A^* \cdot 2a + 1 + R_A^* \cdot a = 0$

$-R_A^* \cdot a + 1 = 0 \qquad R_A^* = R_B^* = \frac{1}{a}$



$0 \le x_1 \le a$

$M_1^*(x) = -R_A^* \cdot x = -\frac{1}{a} \cdot x = -\frac{x}{a}$

$M_1^*(0) = 0$, $M_1^*(a) = -1$

$a \le x_2 \le 2a$

$M_2^*(x) = -R_A^* \cdot x + 1 + R_B^*(x-a) =$
$= -\frac{1}{a} \cdot x + 1 + \frac{1}{a}(x-a) = -\frac{1}{a} \cdot x + 1 + \frac{x}{a} - 1 = 0$

$2a \le x_3 \le 3a$

$M_3^*(x) = 0$

$$\vartheta_B = \frac{1}{EI}\left[\Omega_{1/1}\cdot M_1^*(s_{1/1}) + \Omega_{1/2}\cdot M_1^*(s_{1/2}) + \Omega_{2}\cdot \overset{=0}{M_2^*(s_2)} + M_3\cdot \overset{=0}{M_3^*(s_3)}\right] =$$

$\Omega_{1/1} = -3Pa^2$, $\Omega_{1/2} = -3Pa^2$, $\Omega_2 = -2Pa^2$, $\Omega_3 = +2Pa^2$

$M_1^*(s_{1/1}) = -\frac{1}{2}$, $M_1^*(s_{1/2}) = -\frac{2}{3}$, $M_2^*(s_2) = 0$!, $M_3^*(s_3) = 0$!

$$\vartheta_B = \frac{1}{EI}\left[(-3Pa^2)\cdot\left(-\frac{1}{2}\right) + (-3Pa^2)\left(-\frac{2}{3}\right) + (-2Pa^2)\cdot 0 + 2Pa^2\cdot 0\right] =$$

$$= \frac{1}{EI}\left(\frac{3}{2}Pa^2 + 2Pa^2 + 0 + 0\right) = \frac{7Pa^2}{2EI} = \vartheta_B$$

M-M analytic.

$$\vartheta_B = \frac{1}{EI}\left[\int_0^a M_1(x)\cdot M_1^*(x)\,dx + \int_a^{2a} M_2(x)\cdot \overset{=0}{M_2^*(x)}\,dx + \int_{2a}^{3a} M_3(x)\cdot \overset{=0}{M_3^*(x)}\,dx\right] =$$

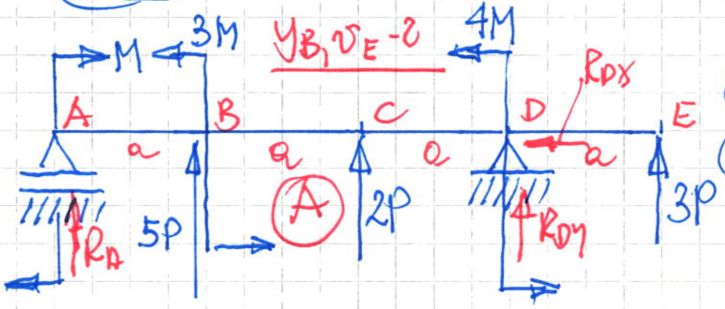
$$= \frac{1}{EI}\int_0^a \left[-3P(2x+a)\right]\left(-\frac{x}{a}\right)dx = \frac{1}{EI}\int_0^a (-6Px - 3Pa)\left(-\frac{x}{a}\right)dx = \frac{1}{EI}\int_0^a \left(+\frac{6Px^2}{a} + \frac{3Pax}{a}\right)dx =$$

$$= \frac{1}{EI}\left(\frac{6P}{a}\frac{x^3}{3} + \frac{3Pa}{a}\frac{x^2}{2}\right)\Bigg|_0^a = \frac{1}{EI}\left(\frac{2Px^3}{a} + \frac{3Px^2}{2}\right)\Bigg|_0^a =$$

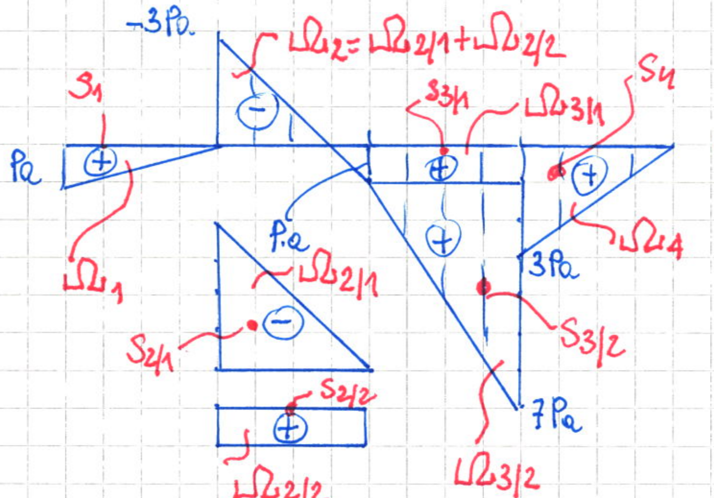
$$= \frac{1}{EI}\left(\frac{2Pa^3}{a} + \frac{3Pa^2}{2}\right) = \frac{1}{EI}\left(2Pa^2 + \frac{3}{2}Pa^2\right) = \frac{7}{2}\frac{Pa^2}{EI} = \vartheta_B$$

Exo 4. $P, a, M = P\cdot a$, $EI = \text{const}$

y_B, v_E - ?



Maxwell-Mohr method (Vereshchagin procedure)



I Static eqs.

① $\sum P_{ix} = -R_{Dx} = 0$

② $\sum P_{iy} = R_A + 5P + 2P + R_{Dy} + 3P = 0$

③ $\sum M_i^D = M + R_A\cdot 3a - 3M + 5P\cdot 2a + 2P\cdot a - 4M - 3P\cdot a = 0$

$Pa + R_A\cdot 3a - 3Pa + 10Pa + 2Pa - 4Pa - 3Pa = 0$

$R_A = -P$

② $\Rightarrow -P + 5P + 2P + R_{Dy} + 3P = 0$

$R_{Dy} = -9P$

II Bending moments

$0 \le x_1 \le a$

$M_1(x) = M + R_A\cdot x = Pa - P\cdot x$

$M_1(0) = Pa$, $M_1(a) = Pa - Pa = 0$

$a \le x_2 \le 2a$

$M_2(x) = M + R_A\cdot x - 3M + 5P(x-a) =$

$= P\cdot a - P\cdot x - 3Pa + 5P\cancel{x} - 5Pa =$

$M_2(x) = 4P\cdot x - 7Pa$

$M_2(a) = 4Pa - 7Pa = -3Pa$

$M_2(2a) = 8Pa - 7Pa = P\cdot a$

$2a \le x_3 \le 3a$

$M_3(x) = M_2(x) + 2P(x-2a) =$

$= 4\cdot P\cdot x - 7\cdot Pa + 2P\cdot x - 4Pa = 6P\cdot x - 11Pa$

$M_3(2a) = Pa$, $M_4(3a) = 7P\cdot a$

$3a \le x_4 \le 4a$

$M_4(x) = M_3(x) - 4M + R_{Dy}\cdot(x-3a) =$

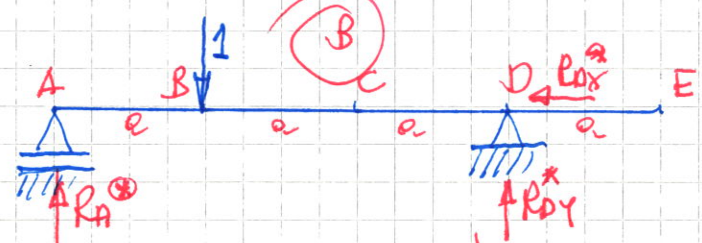
$= 6P\cdot x - 11Pa - 4Pa - 9P(x-3a) =$

$= 6P\cdot x - 11Pa - 4Pa - 9P\cdot x + 27Pa =$

$= -3Px + 12Pa$

$M_4(3a) = -9Pa + 12Pa = 3Pa$

$M_4(4a) = -12Pa + 12Pa = 0$



I static eqs.

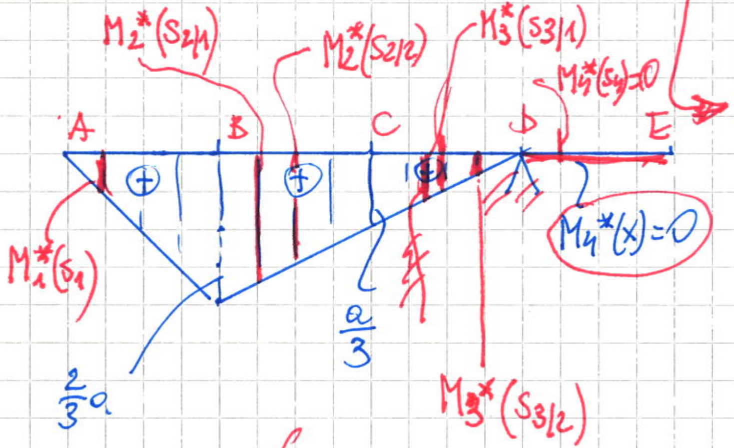
① $\sum P_{ix} = -R_{Dx}^* = 0$

② $\sum P_{iy} = R_A^* - 1 + R_{Dy}^* = 0$

③ $\sum M_i^D = R_A^*\cdot 3a - 1\cdot 2a = 0 \quad |:a$

$R_A^* = \frac{2}{3}$

② $\Rightarrow \frac{2}{3} - 1 + R_{Dy}^* = 0$ $\quad R_{Dy}^* = \frac{1}{3}$



II Bending moments

$0 \le x_1 \le a$

$M_1^*(x) = R_A^*\cdot x = \frac{2}{3}\cdot x$

$M_1^*(0) = 0$, $M_1^*(a) = \frac{2}{3}a$

$a \le x_{2/3} \le 3a$

$M_{2/3}^*(x) = R_A^*\cdot x - 1(x-a) = \frac{2}{3}\cdot x - x + a = -\frac{x}{3} + a$

$M_{2/3}^*(0) = \frac{2}{3}a$, $M_{2/3}^*(2a) = \frac{a}{3}$, $M_{2/3}^*(3a) = 0$

$3a \le x_4 \le 4a \Rightarrow M_4^*(x) = 0$!

$$y_B = \frac{1}{EI}\Big[\Omega_1 \cdot M_1^*(s_1) + \Omega_{2|1} \cdot M_2^*(s_{2|1}) + \Omega_{2|2} \cdot M_2^*(s_{2|2}) + \Omega_{3|1} \cdot M_3^*(s_{3|1}) + \Omega_{3|2} \cdot M_3^*(s_{3|2}) + \Omega_4 \cdot M_4^*(s_4)\Big] = ?$$

$\Omega_1 = \frac{1}{2} Pa^2$, $\Omega_{2|1} = -2Pa^2$, $\Omega_{2|2} = Pa^2$, $\Omega_{3|1} = Pa^2$, $\Omega_{3|2} = 3Pa^2$

$\Omega_4 = \frac{3}{2} Pa^2$

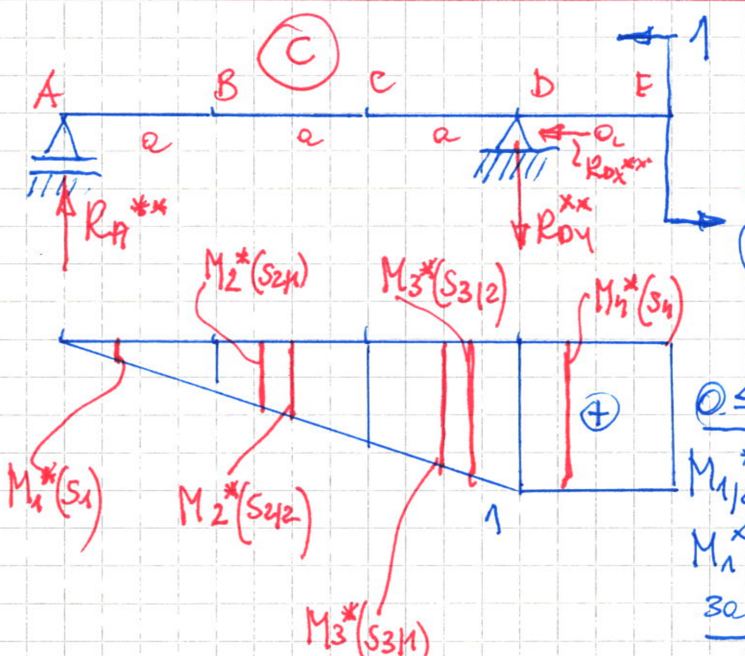
$M_1^*(s_1) = \frac{2}{9}a$, $M_2^*(s_{2|1}) = \frac{5}{9}a$, $M_2^*(s_{2|2}) = \frac{1}{2}a$, $\Omega_3^*(s_{3|1}) = \frac{1}{6}a$

$M_3^*(s_{3|2}) = \frac{1}{9}a$, $M_4^*(s_4) = 0$!

$$y_B = \frac{1}{EI}\Big[\frac{Pa^2}{2}\cdot\frac{2}{9}a - 2Pa^2\cdot\frac{5}{9}a + Pa^2\cdot\frac{1}{2}a + Pa^2\cdot\frac{1}{6}a + 3Pa^2\cdot\frac{1}{9}a + \frac{3}{2}Pa^2\cdot \overset{0}{0}\Big] =$$

$$= \frac{Pa^3}{EI}\Big(\frac{1}{9} \underbrace{- \frac{10}{9} + \frac{1}{2} + \frac{1}{6}}_{0} + \frac{1}{3}\Big) = 0 \ !$$

C



I static eqs.

1) $\sum P_{ix} = -R_{Dx}^{**} = 0$

2) $\sum P_{iy} = R_A^{**} - R_{Dy}^{**} = 0 \Rightarrow R_A^{**} = R_{Dy}^{**}$

3) $\sum M_i^D = R_A^{**} \cdot 3a - 1 = 0$

$R_A^{**} = R_{Dy}^{**} = \frac{1}{3a}$

$0 \le x_{1|2|3} \le 3a$

$M_{1|2|3}^{**}(x) = R_A^{**} \cdot x = \frac{x}{3a}$

$M_1^*(a) = \frac{1}{3}$, $M_1^*(2a) = \frac{2}{3}$, $M_1(3a) = 1$

$3a \le x_4 \le 4a$

$M_4^*(x) = 1$

$$\vartheta_E = \frac{1}{EI}\Big[\Omega_1 \cdot M_1^*(s_1) + \Omega_{2|1} \cdot M_2^*(s_{2|1}) + \Omega_{2|2} \cdot M_2^*(s_{2|2}) + \Omega_{3|1} \cdot M_3^*(s_{3|1}) + \Omega_{3|2} \cdot M_3^*(s_{3|2}) + \Omega_4 \cdot M_4^*(s_4)\Big]$$

$\Omega_1 = \frac{1}{2}Pa^2$, $\Omega_{2|1} = -Pa^2$, $\Omega_{2|2} = Pa^2$, $\Omega_{3|1} = Pa^2$, $\Omega_{3|2} = 3Pa^2$, $\Omega_4 = \frac{3}{2}Pa^2$

$M_1^*(s_1) = \frac{1}{9}\cancel{a}$, $M_2^*(s_{2|1}) = \frac{4}{9}\cancel{a}$, $M_2^*(s_{2|2}) = \frac{1}{2}\cancel{a}$, $M_3^*(s_{3|1}) = \frac{5}{6}\cancel{a}$, $M_3^*(s_{3|2}) = \frac{2}{3}\cancel{a}$

$M_4^*(s_4) = 1$

$$\vartheta_E = \frac{1}{EI}\Big[\frac{Pa^2}{2}\cdot\frac{1}{9}\cancel{a} - Pa^2\cdot\frac{4}{9}\cancel{a} + Pa^2\cdot\frac{1}{2}\cancel{a} + Pa^2\cdot\frac{5}{6}\cancel{a} + 3Pa^2\cdot\frac{2}{3}\cancel{a} + \frac{3}{2}Pa^2\cdot 1\Big] =$$

$$= \frac{Pa^2}{EI}\Big[\frac{1}{18} - \frac{4}{9} + \frac{1}{2} + \frac{5}{6} + 2 + \frac{3}{2}\Big] = \frac{40Pa^2}{9EI}$$