It is obvious that the effect of geometric incompatibility may be useful for regulation of the stresses in the structure. Let us consider a three-hinged arch which is loaded by any fixed load. The bending moments are

$$M(x) = M^0 - Hy,$$

where M^0 is a bending moment in the reference beam. If a tie is fabricated longer then required, then the thrust becomes $H = H_1 + H_2$, where H_1 and H_2 are thrust due to fixed load and errors of fabrication, respectively.

Discussion. For computation of displacement due to the settlement of supports and errors of fabrication, we use the principle of virtual work. The common concept for this principle and for Maxwell–Mohr integral is the concept of generalized coordinate and corresponding generalized unit force.

6.5 Graph Multiplication Method

Graph multiplication method presents most effective way for computation of any displacement (linear, angular, mutual, etc.) of bending structures, particularly for framed structures. The advantage of this method is that the integration procedure according to Maxwell–Mohr integral is replaced by elementary algebraic procedure on two bending moment diagrams in the actual and unit states. This method was developed by Russian engineer Vereshchagin in 1925 and is often referred as the Vereshchagin rule.

Let us consider some portion *AB* which is a part of a bending structure; the bending stiffness, *EI*, within of this portion is constant. The bending moment diagrams for this portion in actual and unit state are M_p and $\bar{M}$. Both diagrams for portion *AB* are presented in Fig. 6.19. In general case, a bending moment diagram M_p in the actual state is bounded by curve, but for special cases it may be bounded by straight line (if a structure is subjected to concentrated forces and/or couples). However, it

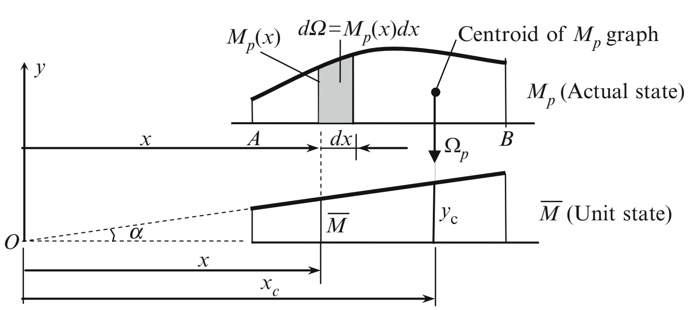

Fig. 6.19 Graph multiplication method. Bending moment diagrams M_p and $\overline{M}$ in actual and unit states

is obvious that in the *unit state* the bending moment diagram $\bar{M}$ is always bounded by the *straight line*. Just this property of unit bending moment diagram allows us to present the Maxwell–Mohr integral for bending systems in the simple form.

Ordinate of the bending moment in actual state at section x is $M_p(x)$. Elementary area of a bending moment diagram in actual condition is $\mathrm{d}\Omega = M_p(x)\mathrm{d}x$. Since $\bar{M} = x\tan\alpha$, then integral in Maxwell–Mohr formula may be presented as (coefficient $1/EI$ by convention is omitted)

$$\int M_p\bar{M}\mathrm{d}x = \int (x\tan\alpha)M_p\mathrm{d}x = \tan\alpha\int x\mathrm{d}\Omega. \tag{6.19}$$

Integral $\int x\mathrm{d}\Omega$ represents the static moment of the area of the bending moment diagram in actual state with respect to axis Oy. It is well known that a static moment may be expressed in terms of total area Ω and coordinate of its centroid x_c by formula $\int x\mathrm{d}\Omega = \Omega_p x_\mathrm{c}$. It is obvious that $x_\mathrm{c}\tan\alpha = y_\mathrm{c}$. Therefore, the Maxwell–Mohr integral may be presented as follows

$$\frac{1}{EI}\int M_p\bar{M}\mathrm{d}x = \frac{\Omega_p y_\mathrm{c}}{EI}. \tag{6.20}$$

The procedure of integration $\int M_p\bar{M}\mathrm{d}x = \Omega_p y_\mathrm{c}$ is called the "multiplication" of two graphs.

The result of multiplication of two graphs, at least one of which is bounded by a straight line (bending moment diagram in unit state), equals to area $\boldsymbol{\Omega}$ of the bending moment diagram $\boldsymbol{M_p}$ in actual state multiplied by the ordinate $\boldsymbol{y_\mathrm{c}}$ from the unit bending moment diagram $\boldsymbol{\bar{M}}$, which is located under the centroid of the $\boldsymbol{M_p}$ diagram.

It should be remembered, that *the ordinate y_c must be taken from the diagram bounded by a straight line.* The graph multiplication procedure (6.20) may be presented by conventional symbol ($\times$) as

$$\Delta_{kp} = \frac{1}{EI}\int M_p\bar{M}_k\mathrm{d}x = \frac{M_p\times\bar{M}_k}{EI}. \tag{6.21}$$

It is obvious that the same procedure may be applicable to calculation of similar integrals, which appear in Maxwell–Mohr integral, i.e., $\int N_p\bar{N}\mathrm{d}x$ and $\int Q_p\bar{Q}\mathrm{d}x$.

If the structure in the actual state is subjected to concentrated forces and/or couples, then both the bending moment diagrams in actual and unit states are bounded by the straight lines (Fig. 6.20a). In this case, the multiplication procedure of two diagrams is commutative. It means that the area Ω could be calculated on any of the two diagrams and corresponding ordinate y_c will be measured from the second one, i.e., $\Omega_1 y_1 = \Omega_2 y_2$. This expression may be expressed in terms of specific ordinates, as presented in Fig. 6.20b.

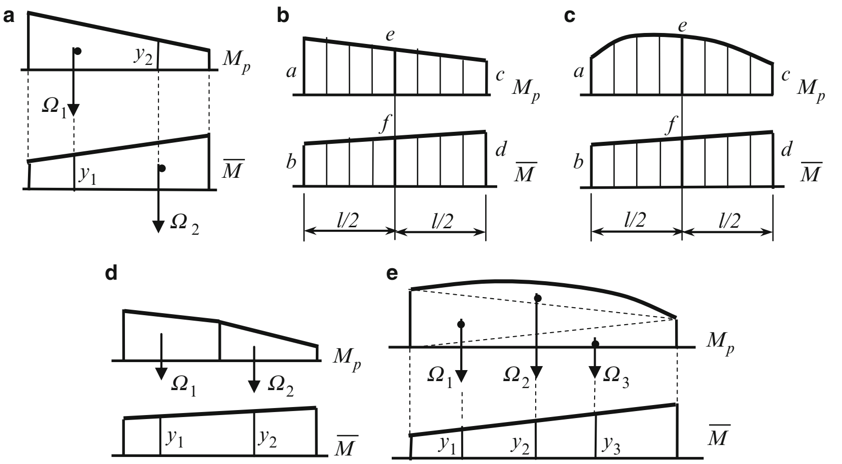

Fig. 6.20 Multiplication of two bending moment diagrams

In this case, the displacement as a result of the multiplication of two graphs may be calculated using two following rules:

1. Trapezoid rule allows calculating the required displacement in terms of *extreme* ordinates

$$\Delta = \frac{l}{6EI}(2ab + 2cd + ad + bc), \tag{6.22}$$

where the crosswise end ordinates has unity coefficients. This formula is precise.

2. Simpson's rule allows calculating the required displacement in terms of *extreme* and *middle* ordinates

$$\Delta = \frac{l}{6EI}(ab + 4ef + cd). \tag{6.23}$$

Equation (6.23) may also be used for calculation of displacements, if the bending moment diagram in the actual condition is bounded by a *curve*line. If the bending moment diagram M_p is bounded by quadratic parabola (Fig. 6.20c), then the result of multiplication of two bending moment diagrams by formula (6.23) is exact; this case occurs if a structure is carrying uniformly distributed load. If the bending moment diagram M_p is bounded by cubic parabola, then the procedure (6.23) leads to the approximate result.

If a graph M_p is bounded by a broken line, then both graphs have to be divided by several portions as shown in Fig. 6.20d. In this case, the result of multiplication of both graphs is

$$\int M_p \bar{M} \mathrm{d}x = \Omega_1 y_1 + \Omega_2 y_2. \tag{6.24}$$

Sometimes it is convenient to subdivide the curved bending moment diagram by a number of "good" shapes, for example in Fig. 6.20e. In this case

$$\int M_p \bar{M} \mathrm{d}x = \Omega_1 y_1 + \Omega_2 y_2 + \Omega_3 y_3. \tag{6.25}$$

Signs rule. According to (6.21), the displacement will be positive, when the area of the diagram M_p and the ordinate y_c of the diagram $\bar{M}$ have the same sign. If ordinates in (6.22) or (6.23) of bending moment diagram for actual and unit states are placed on the *different sides* of the basic line, then result of their multiplication is negative. The positive result indicates that displacement occurs in the direction of applied unit load.

Procedure for computation of deflections by graph multiplication method is as follows:

1. Draw the bending moment diagram M_p for the actual state of the structure.
2. Create a unit state of a structure. For this apply a unit load at the point where the deflection is to be evaluated. For computation of linear displacement we need to apply unit force $P = 1$, for angular displacement to apply unit couple $M = 1$, etc.
3. Draw the bending moment diagram $\bar{M}$ for the unit state of the structure. Since the unit load (force, couple) is dimensionless, then the ordinates of unit bending moment diagram $\bar{M}$ in case of force $F = 1$ and moment $M = 1$ are units of length (m) and dimensionless, respectively.
4. Apply the graph multiplication procedure using the most appropriate form: Vereshchagin rule (6.20), trapezoid rule (6.22), or Simpson's formula (6.23).

Graph multiplication method requires the rapid computation of graph areas of different shapes and determination of the position of their centroid. Table A.1 contains the most typical graphs of bending moment diagrams, their areas, and positions of the centroid. Useful formulas for multiplication of two bending moment diagrams are presented in Table A.2.

Example 6.14. A cantilever beam *AB*, length l, carrying a uniformly distributed load q (Fig. 6.21). Bending stiffness *EI* is constant. Compute (a) the angle of rotation θ_A; (b) the vertical displacement Δ_A at the free end.

Solution. Analysis of the structure starts from construction of bending moment diagram M_p due to given external load. This diagram is bounded by quadratic parabola and maximum ordinate equals $ql^2/2$.

(a) *Angle of rotation at point A*. The unit state presents the same beam subjected to unit couple $M = 1$ at the point where it is required to find angular displacement; direction of this couple is arbitrary (Fig. 6.21a). It is convenient that both unit and actual state and their bending moment diagrams locate one under another.

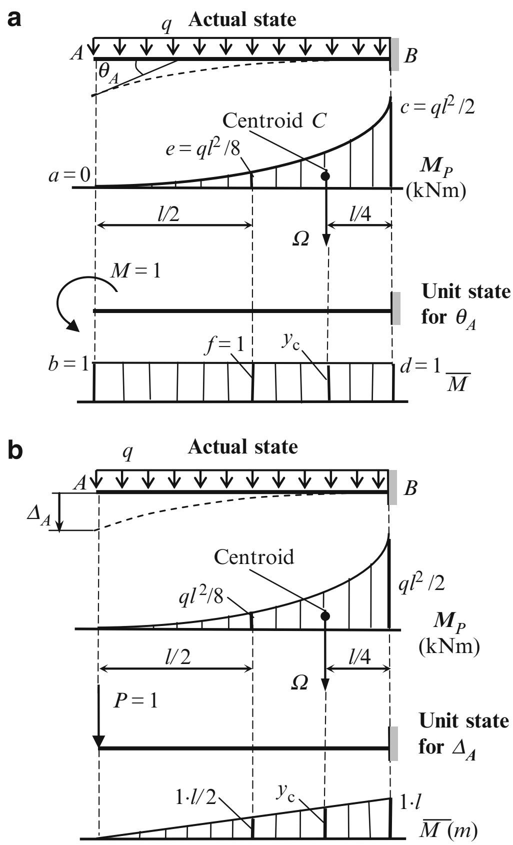

Fig. 6.21 (**a**) Actual state, unit state for θ_A and corresponding bending moment diagrams. (**b**) Actual state, unit state for Δ_A and corresponding bending moment diagrams

The next step is "multiplication" of two bending moment diagrams. The area of square parabola according to Table A.1 is

$$\Omega = \frac{1}{3} \cdot \frac{ql^2}{2} \cdot l.$$

Centroid of this diagram is located on the distance $l/4$ from fixed support. Corresponding ordinate y_c from diagram $\bar{M}$ of unit state is 1. Multiplication procedure is presented in Table 6.5.

This table also contains computation of required displacement using the Simpson rule (6.23). Ordinates a and b are taken from the bending moment diagrams for actual and unit states, respectively, at the left end of a beam (point A); ordinates e and f are taken at the middle of the beam AB, and ordinates c and d at the right end (point B).

Table 6.5 Graph multiplication procedures

Displacement	General formula (6.20) $\Delta = \frac{1}{EI}\Omega y_c$	Simpson rule (6.23) $\Delta = \frac{l}{6EI}(ab + 4ef + cd)$
(a) Angular $\theta_A = \frac{M_p \times \bar{M}}{EI}$	$\theta_A = \frac{1}{EI} \cdot \underbrace{\frac{1}{3}\frac{ql^2}{2}l}_{\Omega} \cdot \underbrace{1}_{y_c} = \frac{ql^3}{6EI}$	$\theta_A = \frac{l}{6EI}\left(\underbrace{0 \cdot 1}_{ab} + \underbrace{4\frac{ql^2}{8} \cdot 1}_{4ef} + \underbrace{\frac{ql^2}{2} \cdot 1}_{cd}\right) = \frac{ql^3}{6EI}$
(b) Linear $\Delta_A = \frac{M_p \times \bar{M}}{EI}$	$\Delta_A = \frac{1}{EI} \cdot \underbrace{\frac{1}{3}\frac{ql^2}{2}l}_{\Omega} \cdot \underbrace{\frac{3}{4} \cdot 1 \cdot l}_{y_c} = \frac{ql^4}{8EI}$	$\Delta_A = \frac{l}{6EI}\left(\underbrace{0 \cdot 0}_{ab} + \underbrace{4\frac{ql^2}{8} \cdot 1 \cdot \frac{l}{2}}_{4ef} + \underbrace{\frac{ql^2}{2} \cdot 1 \cdot l}_{cd}\right) = \frac{ql^4}{8EI}$

(b) *Vertical displacement at point A*. The bending moment diagram M_p for actual state is shown in Fig. 6.21b; this diagram for problems (a) and (b) is same. The unit state presents the same structure with concentrated force $P = 1$, which acts at point A; direction of the unit force is chosen in arbitrary way. The unit state with corresponding bending moment diagram $\bar{M}$ is presented in Fig. 6.21b.

Computation of displacements using Vereshchagin rule in general form and by Simpson rule are presented in Table 6.5.

Discussion:

1. Elastic curve of the beam is shown by dotted line. The tensile fibers for actual and unit states are located above the neutral axis of the beam. Bending moment diagrams are plotted on side of tensile fibers. In the general formula and Simpson rule we use positive sign, because bending moment diagrams for actual and unit states are located on the same side of the basic line. Positive signs in the resulted displacement mean that displacement occurs in the direction of the applied unit load. The units of the ordinates M_p and $\bar{M}$ are (kN m) and (m), respectively.
2. The results, which are obtained by formula (6.20), are precise. Formula (6.23) is approximate one, but for the given problem it leads to the exact result, because the beam is loaded by *uniformly* distributed load, the bending moment diagram presents quadratic parabola, and total order of curves presenting two bending moment diagrams in the actual and unit states is equal to three. If the total orders are more than three, then formula (6.23) leads to the approximate result.
3. The reader is invited to solve the problems above by double integration method, initial parameters method, conjugate beam method, Castigliano theorem, Maxwell–Mohr integral, compare their effectiveness with graph multiplication method, and make personal conclusion about its proficient.

Example 6.15. Design diagram of symmetrical nonuniform simply supported beam of length l is shown in Fig. 6.22. Bending stiffness equals EI for segments AD and EB; while kEI for segment DE; parameter k is any positive number. The beam is carrying force P. Determine the vertical displacement of point C.

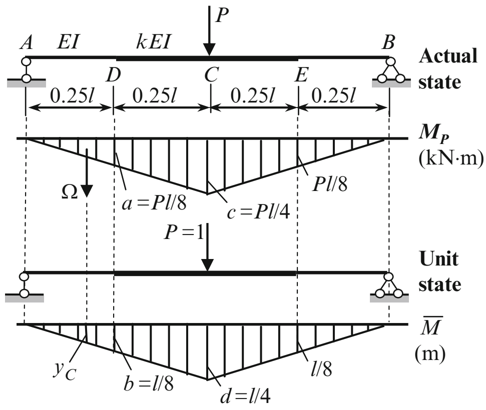

Fig. 6.22 Design diagram of the beam and bending moment diagrams for actual and unit states

Solution. Bending moment diagrams in actual and unit states are presented in Fig. 6.24. For computation $\Delta_C = (M_p \times \bar{M})/EI$, we have to subdivide bending moment diagrams on the parts, within which the bending stiffness is constant. These parts for the left half of the beam are $AD = l/4$ and $DC = l/4$. The Vereshchagin rule for multiplication of diagrams M_p and $\bar{M}$ within portion AD leads to the following result

$$\Delta_{C1} = \frac{1}{EI} \cdot \underbrace{\frac{1}{2} \cdot \frac{Pl}{8} \cdot \frac{l}{4}}_{\Omega} \cdot \underbrace{\frac{2}{3} \cdot \frac{l}{8}}_{y_c} = \frac{Pl^3}{768EI}. \tag{a}$$

For portion CD the trapezoid rule is applied. According to (6.22) we get

$$\Delta_{C2} = \frac{l/4}{6 \cdot k\,EI}\left[\underbrace{2 \cdot \frac{Pl}{8} \cdot \frac{l}{8}}_{2ab} + \underbrace{2\frac{Pl}{4} \cdot \frac{l}{4}}_{2cd} + \underbrace{\frac{Pl}{8} \cdot \frac{l}{4}}_{ad} + \underbrace{\frac{Pl}{4} \cdot \frac{l}{8}}_{cb}\right] = \frac{7}{768}\frac{Pl^3}{k\,EI}. \tag{b}$$

Finally, the vertical displacement at C becomes

$$\Delta_C = 2\left(\frac{Pl^3}{768EI} + \frac{7}{768}\frac{Pl^3}{k\,EI}\right) = \frac{Pl^3}{48EI}\eta,$$

where $\eta = (1/8) + (7/8k)$; factor 2 takes into account symmetrical part CEB of the beam. If $k = 1$, then $\eta = 1$.

Example 6.16. A portal frame is subjected to horizontal force P as shown in Fig. 6.23. The bending stiffness for each member is shown on design diagram. Calculate (a) the horizontal displacement at the rolled support B and (b) the angle of rotation at the same point B.

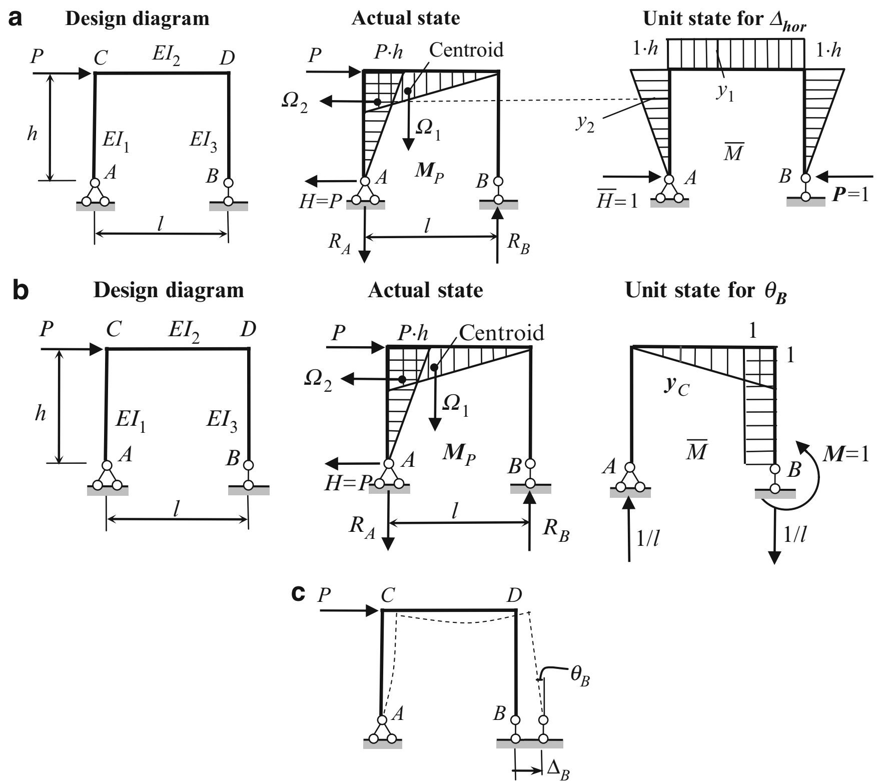

Fig. 6.23 (**a**) Portal frame. Actual and unit states and corresponding bending moment diagrams. (**b**) Portal frame. Actual and unit states and corresponding bending moment diagrams. (**c**) Design diagram and elastic curve for portal frame

Solution. As usual, the analysis starts from construction of bending moment diagram in actual state.

Reactions of supports are $H = P$, $R_A = R_B = Ph/l$. The real directions of reactions are shown in Fig. 6.23a. The tensile fibers on elements *CD* and *AC* are located below and right from the neutral lines of the elements, respectively. Bending moment ordinates at point C for vertical and horizontal members are *Ph*.

(a) *Horizontal displacement at B.* For required displacement Δ_{hor}, the unit state presents the same frame with horizontal force $P = 1$, which is applied at point B. Direction of the unit force is chosen in arbitrary way. Only horizontal reaction $H = 1$ is induced. The tensile fibers are located outdoor of the frame. The bending moments at rigid joints C and D and within cross bar equal to $1 \cdot h$.

Multiplication of the bending moment diagrams should be performed for members *AC*, *CD*, and *DB* separately. For horizontal member *CD*, the area of the bending moment diagram in actual state is $\Omega_1 = (1/2)Ph \cdot l$ and corresponding ordinate from unit state is $y_1 = 1 \cdot h$. We assume that horizontal portion *CD* of both bending moment diagrams is located one under the other and vertical portion *AC* is located one besides the other. For vertical member *AC*, the area of the bending moment diagram in actual state is $\Omega_2 = (1/2)h \cdot Ph$. Corresponding ordinate from unit bending moment diagram is $y_2 = (2/3) \cdot 1 \cdot h$ (Fig. 6.23a). Using Vereshchagin rule and taking into account the different flexural rigidities for the vertical and horizontal members, we find the required displacement:

$$\Delta_B = \sum \frac{1}{EI} \int_0^s M_p \bar{M} \mathrm{d}s = \underbrace{-\frac{1}{EI_1} \cdot \underbrace{\frac{1}{2}h \cdot Ph}_{\Omega} \cdot \underbrace{\frac{2}{3} \cdot 1 \cdot h}_{y_c}}_{AC \text{ element}} \underbrace{- \frac{1}{EI_2} \cdot \underbrace{\frac{1}{2}Ph \cdot l}_{\Omega} \cdot \underbrace{1 \cdot h}_{y_c}}_{CD \text{ element}}$$

$$= -\frac{Ph^3}{3EI_1} - \frac{Plh^2}{2EI_2}. \tag{a}$$

Each term of the expression for horizontal displacement has negative sign, because the bending moment diagrams for actual and unit states are located on different sides of the basic line of the frame. The result of multiplication of diagrams within the vertical member *BD* equals to zero, since in actual state the bending moments within the member *BD* are zeros. A final negative sign means that assumed unit force produces a negative work along the real horizontal displacement Δ_B.

(b) *Angle of rotation at B*. The bending moment diagram for actual state is shown in Fig. 6.23b; this diagram for problems (a) and (b) is same. For required displacement θ_B, the unit state presents the same structure with concentrated couple $M = 1$, which acts at point *B*; direction of the unit couple is chosen in arbitrary way. In the unit state, only vertical reactions $1/l$ arise. The extended fibers are located indoor of the frame. The centroid and area Ω of bending moment diagram in actual state and corresponding ordinate y_c from bending moment diagram for unit state are shown in Fig. 6.23b.

The result of multiplication of diagrams within two vertical members *AC* and *BD* equals to zero. Indeed, for these portions the procedure $M_p \times \bar{M} = 0$ because $\bar{M} = 0$ for member *AC*, and $M_p = 0$ for member *BD*. For member *CD*, the Vereshchagin rule leads to the following result

$$\theta_B = \sum \frac{1}{EI} \int_0^s M_p \bar{M} \mathrm{d}s = \underbrace{\frac{1}{EI_2} \cdot \underbrace{\frac{1}{2}Ph \cdot l}_{\Omega} \cdot \underbrace{\frac{1}{3} \cdot 1}_{y_c}}_{CD \text{ element}} = \frac{Plh}{6EI_2}. \tag{b}$$

The positive sign is adopted because the bending moment diagrams for actual and unit states are located on one side of the basic line *CD* of the frame. A final positive sign means that assumed unit couple produce a positive work along the real angular displacement θ_B. Or by other words, the actual direction of angular displacement coincides with the chosen direction for unit couple M, i.e., the section at support *B* rotates counterclockwise.

Discussion. In actual state, the bending moment along the right vertical bar does not arise; as a result, multiplication of bending moment diagrams for this element for both problems (a) and (b) equals to zero. Therefore, the final result for problems (a) and (b) does not contain a term with stiffness EI_3. This happens because in actual state the element *BD* does not subjected to bending, i.e., this member has displacement as absolutely rigid body. Elastic curve of the frame is shown in Fig. 6.23c.

6.6 Elastic Loads Method

Elastic load method allows *simultaneous* computation of displacements for *set* of points of a structure. This method is based on conjugate beam method. The method especially effective for computation of displacements for set of joints of the truss chord; for trusses this method leads to the precise results.

Elastic loads W are fictitious loads which are applied to the conjugate structure. Bending moments of the conjugate structure and displacements of the real structure at the point of application of elastic loads coincide. A final expression for elastic load at joint n of the truss may be calculated by formula

$$W_n = \sum \frac{\overline{N}_n \cdot N_p \cdot l}{EA} \tag{6.26}$$

This formula uses the following notation: N_p is internal forces due to given load and $\bar{N}_n$ is internal forces in all members of the truss in the unit state.

The right part of the formula (6.26) is similar to formula (6.12), however left part of 6.26 is elastic load, while in Maxwell-Mohr formula - left part is displacement.

Computation of displacements procedure is as follows:

1. Calculate the axial forces N_p in all elements of the truss caused by given load.
2. Calculate the elastic load at a joint n. For this:
    a. Show a fictitious truss. If a real truss is simply supported then the fictitious truss is also simply supported.
    b. Apply two unit couples $M = 1$ to members, which are adjacent to the joint n. Present each couple using forces $F_{n-1} = 1/d_{n-1}$ for span d_{n-1} and $F_n = 1/d_n$ for span d_n, as shown in Fig. 6.24.
    c. Calculate the axial forces $\bar{N}_n$ in all elements of the truss caused by forces in Fig. 6.24.
    d. Calculate the elastic load W_n at the joint n by formula (6.26).
3. Calculate the elastic loads W for remaining joints of the truss chord, as explained in pos. 2.
4. Show the fictitious simply supported beam subjected to all elastic loads W. If the elastic load is positive, then it should be directed downward, i.e., in the same direction as the adjacent forces of neighboring couples.