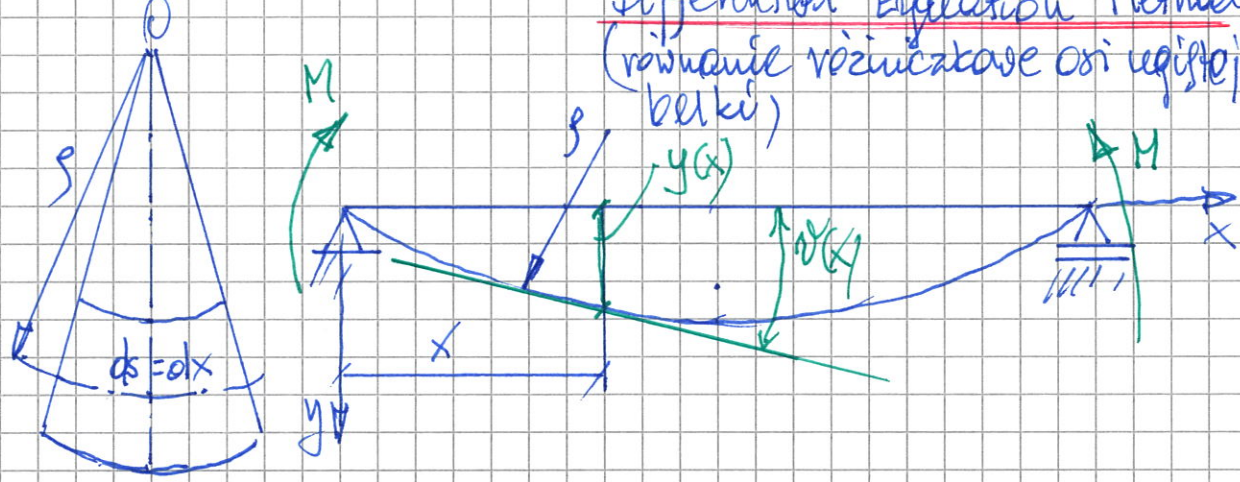
Bending Deflection – Differential Equation Method
(równanie różniczkowe osi ugiętej belki)



Hence, we obtain following expression for the curvature of the beam:

$$\frac{1}{\rho} = \frac{-M}{EI_z}$$

It is known that the ~~radius~~ curvature is given by

$$\frac{1}{\rho} = \pm \frac{y''}{\sqrt{\left(1+(y')^2\right)^3}}; \quad tg\vartheta = y' = \frac{dy}{dx}; \quad y'' = \frac{d^2y}{dx^2}$$

Assuming that the deflection of the beam is sufficiently small, we can neglect the first derivative y'.
Then the differential equation of the elastic line (axis) of beam can be written as follow:

$$\frac{-M}{EI_z} = \pm \frac{y''}{\sqrt{\left(1+(y')^2\right)^3}}$$

ϑ — small

$$tg\vartheta = \frac{dy}{dx} \approx 0$$

$$\frac{-M}{EI_z} = \frac{d^2y(x)}{dx^2} \Rightarrow \boxed{EI\frac{d^2y(x)}{dx^2} = -M}$$

$$EI\frac{d^2y}{dx^2} = -M(x)$$

$$EI\frac{dy}{dx} = -\int M(x)dx + C$$

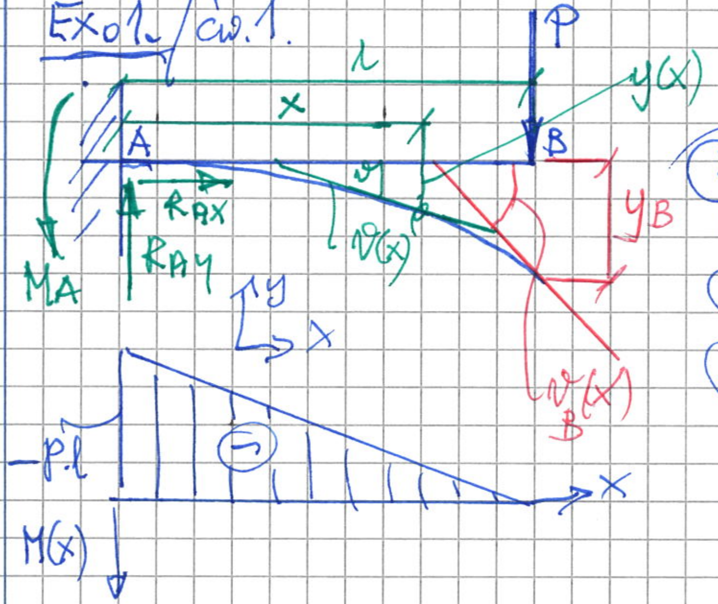
$$EI\,y = -\int\left[\int M(x)dx\right] + Cx + D$$

C, D – from boundary conditions

but $\frac{d^2M(x)}{dx^2} = -q(x) \Rightarrow EI\frac{d^4y(x)}{dx^4} = q(x)$

boundary of beam section	boundary conditions	number of conditions
fixed end	$y=0,\ \vartheta=0$	2
pinned	$y=0$	1
roller	$\vartheta_L=\vartheta_r,\ y_L=0,\ y_r=0$	3
hinge	$y_L=y_r$	1
place of force or moment action, limit of continous load	$\vartheta_L=\vartheta_r,\ y_L=y_r$	2

Ex 1 / ćw. 1



$P,\ l,\ EI = const$

$\vartheta(x),\ y(x),\ \vartheta_B,\ y_B$ - ?

I static conditions

1. $\sum P_{ix} = R_{AX} = 0$
2. $\sum P_{iy} = R_{AY} - P = 0 \Rightarrow R_{AY} = P$
3. $\sum M_i^A = -M_A + P \cdot l = 0$

$M_A = P \cdot l$

$M(x) = -M_A + R_{AY} \cdot x = -P\cdot l + P \cdot x$
$= P(x - l)$
$M(0) = P \cdot l$
$M(l) = 0$

II diff. equation

$EI\frac{d^2y}{dx^2} = -M(x) = P\cdot l - P\cdot x$

$EI\frac{dy}{dx} = Plx - \frac{Px^2}{2} + C$

$EI\, y = \frac{Plx^2}{2} - \frac{Px^3}{6} + C\cdot x + D$

C, D - ?

III Boundary conditions

1. $x = 0,\ \vartheta = 0$
2. $x = 0,\ y = 0$

from (1) $\quad P \cdot l \cdot 0 - \frac{P \cdot 0^2}{2} + C = 0 \Rightarrow C = 0$

from (2) $\quad \frac{P \cdot l \cdot 0^2}{2} - \frac{P \cdot 0^3}{6} + 0 \cdot 0 + D = 0 \Rightarrow D = 0$

last

$$\boxed{EI\frac{dy}{dx} = P \cdot l \cdot x - \frac{Px^2}{2}}$$

$$\boxed{EI\,y = \frac{Plx^2}{2} - \frac{Px^3}{6}}$$

for B $\quad x = l$

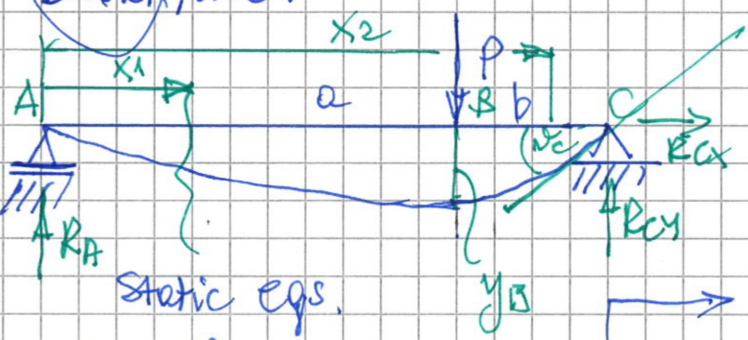
$$EI\frac{dy(l)}{dx} = Pl \cdot l - \frac{P \cdot l^2}{2} = Pl^2 - \frac{Pl^2}{2} = \frac{Pl^2}{2}$$

$$\vartheta_B = \frac{dy(x=l)}{dx} = \frac{Pl^2}{2EI}$$

$$EI\,y(x=l) = \frac{Pl \cdot l^2}{2} - \frac{P \cdot l^3}{6} = \frac{Pl^3}{2} - \frac{Pl^3}{6} = \frac{Pl^3}{3}$$

$$y(x=l) = \frac{Pl^3}{3EI}$$

Ex. 2, lcw 2.



$P, a, b, EI = \text{const}$

$\vartheta(x), y(x), y_B, \vartheta_C - ?$

Static eqs.

(1) $\Sigma P_{ix} = R_{Cx} = 0$

(2) $\Sigma P_{iy} = R_A - P + R_{Cy} = 0$

(3) $\Sigma M_i^C = R_A(a+b) - P \cdot b = 0$

$\rightarrow R_A = P\frac{b}{a+b}$

from (2) $\quad P\frac{b}{a+b} - P + R_{Cy} = 0$

$P \cdot b - P(a+b) + R_{Cy}(a+b) = 0$

$\cancel{P \cdot b} - P \cdot a - \cancel{P \cdot b} + R_{Cy}(a+b)$

$R_{Cy} = P\frac{a}{a+b}$

$0 \le x_1 \le a$

$$M_1(x) = R_A \cdot x = P\frac{b}{a+b}\cdot x$$

$$EI\frac{d^2y(x)}{dx^2} = -M_1(x) = -P\frac{b}{a+b}\cdot x$$

$$EI\frac{dy}{dx} = -P\frac{b}{a+b}\frac{x^2}{2} + C_1 \quad **$$

$$EI\,y(x) = -P\frac{b}{a+b}\frac{x^3}{6} + C_1 x + D_1 \quad *$$

$a \le x_2 \le a+b$

$$M_2(x) = R_A\cdot x - P(x-a) = R_A\cdot x - P\cdot x + P\cdot a$$

$$EI\frac{d^2y(x)}{dx^2} = -M_2(x) = -R_A\cdot x + P\cdot x - P\cdot a = (P-R_A)x - P\cdot a$$

$$EI\frac{dy}{dx} = (P-R_A)\frac{x^2}{2} - P\cdot a\cdot x + C_2 = \left(P - P\frac{b}{a+b}\right)\frac{x^2}{2} - P\cdot a\cdot x + C_2 \quad ***$$

$$EI\,y = \left(P - P\frac{b}{a+b}\right)\frac{x^3}{6} - Pa\frac{x^2}{2} + C_2\cdot x + D_2 \quad 4*$$

C_1, C_2, D_1, D_2 – ?

Boundary conditions

① $x=0$, $y=0$ $*$
② $x=a$, $v_L = v_R$ $** = ***$
③ $x=a$, $y_L = y_R$ $* = 4*$
④ $x=a+b$, $y=0$ $4*$

$\Rightarrow C_1, C_2, D_1, D_2$

$y_B \Rightarrow *$ or $4*$
$v_C \Rightarrow ***$

① $\Rightarrow$ $-P\cancel{\frac{b}{a+b}}\cancel{\frac{0^3}{6}} + C_1\cdot 0 + D_1 = 0 \Rightarrow D_1 = 0$

② $\Rightarrow$ $-P\frac{b}{a+b}\frac{a^2}{2} + C_1 = \left(P - P\frac{b}{a+b}\right)\frac{a^2}{2} - P\cdot a\cdot a + C_2$

$$-P\frac{a^2 b}{a+b} + C_1 = \frac{Pa + \cancel{Pb} - \cancel{Pb}}{a+b}\frac{a^2}{2} - Pa^2 + C_2$$

$$-P\frac{a^2 b}{a+b} + C_1 = \frac{Pa^3}{2(a+b)} - Pa^2 + C_2$$

$$C_1 = \frac{Pa^3}{2(a+b)} + \frac{Pa^2 b}{a+b} - Pa^2 + C_2 = \frac{Pa^3 + 2Pa^2 b}{2(a+b)} - Pa^2 + C_2$$

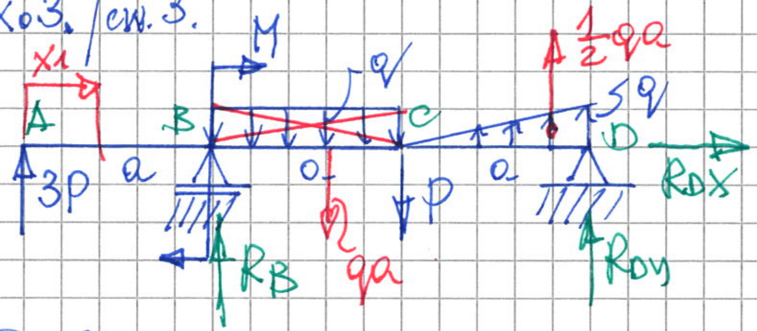
$$= \frac{\cancel{Pa^2}\; Pa^3 + \cancel{2Pa^2b} - 2Pa^3 - \cancel{2Pa^2b}}{2(a+b)} + C_2$$

$$C_1 = \frac{-Pa^3}{2(a+b)} + C_2$$

③ $\Rightarrow$ $-P\frac{b}{a+b}\frac{a^3}{6} + C_1 a + D_1 = \left(P - P\frac{b}{a+b}\right)\frac{a^3}{6} - Pa\frac{a^2}{2} + C_2 a + D_2$

④ $\Rightarrow$ $\cancel{EI}\left(P - P\frac{b}{a+b}\right)\frac{(a+b)^3}{6} - Pa\frac{(a+b)^2}{2} + C_2(a+b) + D_2 = 0$

Ex. 3. / ćw. 3.



$q, a, P = qa, M = qa^2$

$EI = const$

$\vartheta(x), y(x), \vartheta_A, y_C - ?$

(I) Static eqs.

① $\sum P_{ix} = R_{Dx} = 0$

② $\sum P_{iy} = 3P + R_B - qa - P + \frac{1}{2}qa + R_{Dy} = 0$

③ $\sum M_i^D = 3P \cdot 3a + R_B \cdot 2a + M - qa \cdot \frac{3}{2}a - P \cdot a + \frac{1}{2}qa \cdot \frac{1}{3}a = 0$

$\Rightarrow R_B, R_{Dy}$

$0 \le x_1 \le a$

$M_1(x) = 3P \cdot x$

$EI\frac{d^2y}{dx^2} = -M_1(x) = -3P \cdot x$

$EI\frac{dy}{dx} = -3P\frac{x^2}{2} + C_1$ (*)

$EI\, y(x) = -3P\frac{x^3}{6} + C_1 x + D_1$ (**)

$a \le x_2 \le 2a$

$M_2(x) = 3P \cdot x + R_B(x-a) + M - \frac{q(x-a)^2}{2}$

$M_2(x) = 3P \cdot x + R_B \cdot x - R_B \cdot a + M - \frac{q}{2}(x^2 - 2ax + a^2)$

$EI\frac{d^2y}{dx^2} = -3P \cdot x - R_B \cdot x + R_B a - M + \frac{q}{2}(x^2 - 2ax + a^2)$

$EI\frac{dy}{dx} = -3P\frac{x^2}{2} - R_B\frac{x^2}{2} + R_B a \cdot x - M \cdot x + \frac{q}{2}\left(\frac{x^3}{3} - ax^2 + a^2 x\right) + C_2$ (***)

$EI\, y(x) = -3P\frac{x^3}{6} - R_B\frac{x^3}{6} + R_B a\frac{x^2}{2} - M\frac{x^2}{2} + \frac{q}{2}\left(\frac{x^4}{12} - a\frac{x^3}{3} + a^2\frac{x^2}{2}\right) + C_2 x + D_2$ (4*)

$2a \le x_3 \le 3a$

$M_3(x) = 3P \cdot x + M + R_B(x-a) - qa\left(x - \frac{3}{2}a\right) - P(x-2a) + \frac{q(x-2a)^3}{6a}$?

$\frac{q(x)}{x-2a} = \frac{q}{3a-2a}$

$\frac{q(x)}{x-2a} = \frac{q}{a}$

$q(x) = q\frac{(x-2a)}{a}$

$\frac{1}{2}q(x)(x-2a) \cdot \frac{1}{3}(x-2a) =$

$= \frac{1}{2}q\frac{(x-2a)}{a} \cdot (x-2a) \cdot \frac{1}{3}(x-2a) =$

$= q\frac{(x-2a)^3}{6a}$

$$EI\frac{d^2y(x)}{dx^2} = -M_3(x) = -3P\,x - M - R_B(x-a) + qa\left(x-\frac{3}{2}a\right) + P(x-2a) - q\,\frac{(x-2a)^3}{6a}$$

$$EI\frac{d^2y(x)}{dx^2} = -3P\cdot x - M - R_B\cdot x + R_B\cdot a + qa\cdot x - \frac{3}{2}qa^2 + P\cdot x - 2Pa - \frac{q}{6a}\left(x^3 - 3x^2\cdot 2a + 3x\cdot 4a^2 - 8a^3\right)$$

$$EI\frac{dy}{dx} = -3P\frac{x^2}{2} - Mx - R_B\frac{x^2}{2} + R_B\cdot a\cdot x + qa\frac{x^2}{2} - \frac{3}{2}qa^2x + P\frac{x^2}{2} - 2Pa\cdot x - \frac{q}{6a}\left(\frac{x^4}{4} - 6a\frac{x^3}{3} + 12a^2\frac{x^2}{2} - 8a^3\cdot x\right) + C_3 \quad (5*)$$

$$EI\,y(x) = -3P\frac{x^3}{6} - M\frac{x^2}{2} - R_B\frac{x^3}{6} + R_Ba\frac{x^2}{2} + qa\frac{x^3}{6} - \frac{3}{2}qa^2\frac{x^2}{2} + P\frac{x^3}{6} - 2Pa\frac{x^2}{2} - \frac{q}{6a}\left(\frac{x^5}{20} - \frac{ax^4}{2} + 2a^2x^3 - 4a^3x^2\right) + C_3\cdot x + D_3 \quad (6*)$$

$$\underbrace{C_1, C_2, C_3, D_1, D_2, D_3}_{6} - 2$$

Boundary conditions

(1)	$x=a$	$\vartheta_L = \vartheta_R$	(*) = (**)
(2)	$x=a$	$y_L \Rightarrow 0$	(*) (**) (*)
(3)	$x=a$	$y_R \Rightarrow 0$	(4*)
(4)	$x=2a$	$\vartheta_L = \vartheta_R$	(**) = (5*)
(5)	$x=2a$	$y_L = y_R$	(4*) = (6*)
(6)	$x=3a$	$y=0$	(6*)