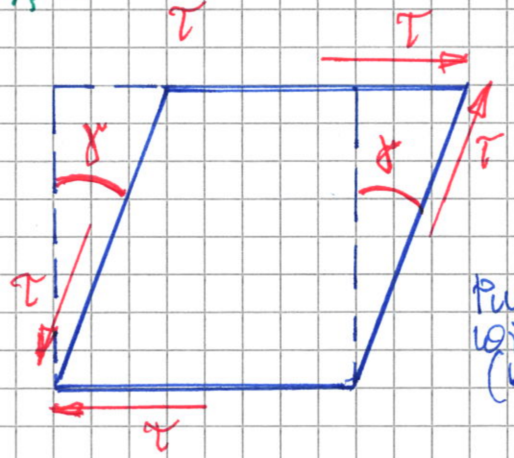
Pure shear
Czyste ścinanie

- $\sigma = 0$, $\tau \neq 0$
- instead of square, we have a rhombus (zamiast kwadratu mamy romb)
- angle γ is the measure of deformation/strain (miarą odkształcenia jest kąt γ)
- γ - shear strain (tangential strain) (odkształcenie postaciowe/styczne)

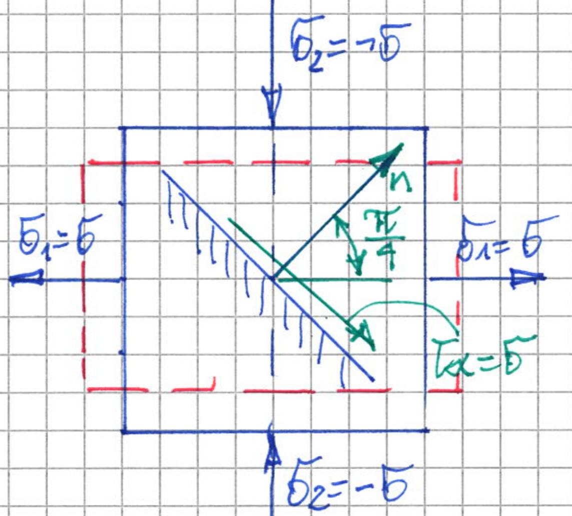


Pure shear state can be realized with the help of two normal (main) stresses σ_1 and σ_2

$$|\sigma_1| = |\sigma_2|$$

(stan czystego ścinania można uzyskać za pomocą dwóch głównych naprężeń normalnych σ_1 i σ_2

$$|\sigma_1| = |\sigma_2|$$

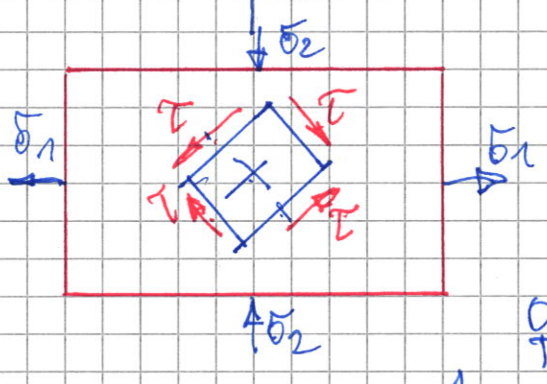


$$\alpha = \frac{\pi}{4}$$

$$\sigma_\alpha = \frac{\sigma_1+\sigma_2}{2} + \frac{\sigma_1-\sigma_2}{2}\cos 2\alpha = \frac{\sigma-\sigma}{2} + \frac{\sigma-(-\sigma)}{2}\cdot 0 = 0 \;!$$

$$\tau_\alpha = \frac{\sigma_1-\sigma_2}{2}\sin 2\alpha = \frac{\sigma-(-\sigma)}{2}\cdot 1 = \sigma$$

$$\sigma_\alpha = 0 \quad \tau_\alpha = \sigma$$



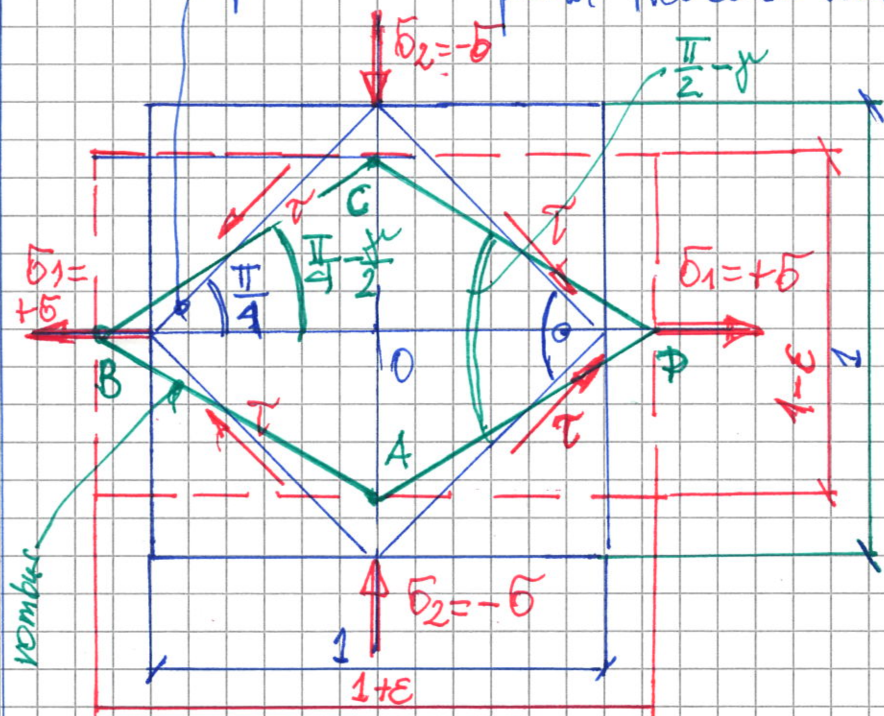
$$\sigma\left(\alpha+\frac{\pi}{2}\right) = \sigma_{\frac{3\pi}{4}} = 0$$

$$\tau\left(\alpha+\frac{\pi}{2}\right) = \tau_{\frac{3\pi}{4}} = -\sigma$$

Only shear ~~strains~~ stresses (pure shear)
Tylko naprężenia styczne

Hooke's law for shear

prawo Hooke'a dla ścinania



$$\varepsilon_1=\frac{1}{E}\sigma_1-\nu\cdot\frac{1}{E}\sigma_2=\frac{1}{E}\left[\sigma_1-\nu(-\sigma_2)\right]=\frac{1}{E}(\sigma+\nu\sigma)=\frac{1+\nu}{E}\sigma$$

$$\varepsilon_2=\frac{1}{E}(\sigma_2-\nu\sigma_1)=\frac{1}{E}\left[(-\sigma)-\nu\sigma\right]=-\frac{1+\nu}{E}\cdot\sigma$$

$$|\varepsilon_1|=|\varepsilon_2|=\varepsilon$$

$$\angle ABC=\angle ADC=\frac{\pi}{2}-\gamma$$

$$\angle OBC=\frac{\pi}{4}-\frac{\gamma}{2}$$

$$tg\left(\frac{\pi}{4}-\frac{\gamma}{2}\right)=\frac{\frac{1-\varepsilon}{2}}{\frac{1+\varepsilon}{2}}=\frac{1-\varepsilon}{1+\varepsilon}$$

from trigonometry

but $tg\left(\frac{\pi}{4}-\frac{\gamma}{2}\right)=\frac{tg\frac{\pi}{4}-tg\frac{\gamma}{2}}{1+tg\frac{\pi}{4}\cdot tg\frac{\gamma}{2}}=\frac{1-tg\frac{\gamma}{2}}{1+tg\frac{\gamma}{2}}=$

$=\frac{1-\frac{\gamma}{2}}{1+\frac{\gamma}{2}}$ $\left(tg\frac{\gamma}{2}\cong\frac{\gamma}{2}\right.$ for small angles$\left.\right)$!

$$\boxed{tg\left(\frac{\pi}{4}-\frac{\gamma}{2}\right)=\frac{1-\frac{\gamma}{2}}{1+\frac{\gamma}{2}}=\frac{1-\varepsilon}{1+\varepsilon}}$$

$$\gamma=2\varepsilon \qquad \boxed{\varepsilon=\frac{\gamma}{2}}$$

for pure share
(dla czystego ścinania)

$$\boxed{\tau = \sigma}$$

$$\varepsilon = |\varepsilon_1| = |\varepsilon_2| = \frac{1+\nu}{E}\cdot\sigma = \frac{1+\nu}{E}\cdot\tau = \frac{\gamma}{2} \Rightarrow$$

$$\gamma = \frac{2(1+\nu)}{E}\cdot\tau \qquad \gamma = \frac{\tau}{\frac{E}{2(1+\nu)}};$$

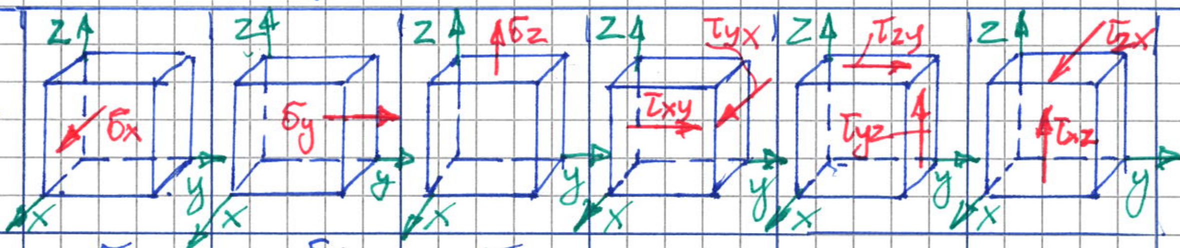
$$\cancel{\frac{E}{2(1+\nu)}} \quad \boxed{\frac{E}{2(1+\nu)} = G}$$

G – Kirchhoff's moduler
Moduł Kirchhoffa

$$\boxed{\gamma = \frac{\tau}{G}} \Leftrightarrow$$ Hooke's law for shear
Prawo Hooke'a dla ścinania

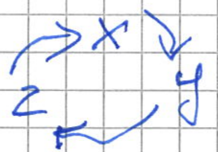
$\varepsilon = \frac{\sigma}{E}$ – prawo Hooke'a dla rozciągania

The generalized Hooke's law
Uogólnione prawo Hooke'a



strain	σ_x	σ_y	σ_z	τ_{xy}	τ_{yz}	τ_{zx}
ε_x	$\frac{\sigma_x}{E}$	$-\nu\frac{\sigma_y}{E}$	$-\nu\frac{\sigma_z}{E}$	0	0	0
ε_y	$-\nu\frac{\sigma_x}{E}$	$\frac{\sigma_y}{E}$	$-\nu\frac{\sigma_z}{E}$	0	0	0
ε_z	$-\nu\frac{\sigma_x}{E}$	$-\nu\frac{\sigma_y}{E}$	$\frac{\sigma_z}{E}$	0	0	0
γ_{xy}	0	0	0	$\frac{\tau_{xy}}{G}$	0	0
γ_{yz}	0	0	0	0	$\frac{\tau_{yz}}{G}$	0
γ_{zx}	0	0	0	0	0	$\frac{\tau_{zx}}{G}$

summing up individual lines
sumując poszczególne wiersze



$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right] \qquad \gamma_{xy} = \frac{\tau_{xy}}{G}$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_z + \sigma_x)\right] \qquad \gamma_{yz} = \frac{\tau_{yz}}{G}$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right] \qquad \gamma_{zx} = \frac{\tau_{zx}}{G}$$

after simple transformations

$$\sigma_x = \frac{E}{1+\nu}\left[\varepsilon_x + \frac{\nu}{1-2\nu}(\varepsilon_x + \varepsilon_y + \varepsilon_z)\right]; \quad \tau_{xy} = G\cdot\gamma_{xy}$$

$$\sigma_y = \frac{E}{1+\nu}\left[\varepsilon_y + \frac{\nu}{1-2\nu}(\varepsilon_x + \varepsilon_y + \varepsilon_z)\right]; \quad \tau_{yz} = G\cdot\gamma_{yz}$$

$$\sigma_z = \frac{E}{1+\nu}\left[\varepsilon_z + \frac{\nu}{1-2\nu}(\varepsilon_x + \varepsilon_y + \varepsilon_z)\right]; \quad \tau_{zx} = G\cdot\gamma_{zx}$$

for major directions
dla kierunków głównych

$$\tau_{xy} = \tau_{yz} = \tau_{zx} = 0$$

for planar stress state

$$\sigma_z = 0, \ \tau_{yz} = 0, \ \tau_{xz} = 0$$

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y); \quad \varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x); \quad \gamma_{xy} = \frac{\tau_{xy}}{G}$$

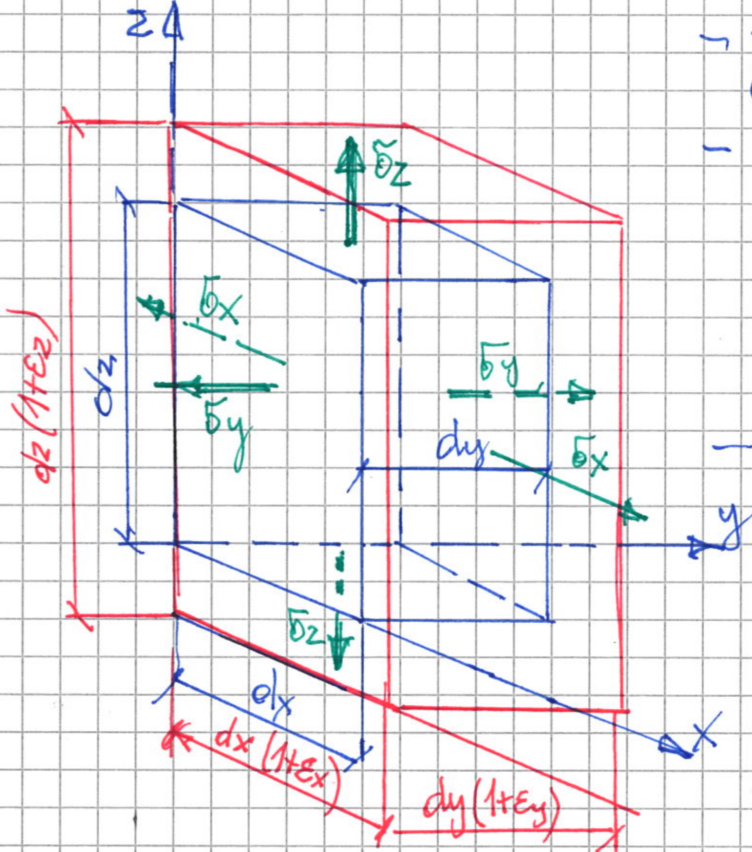
$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y); \quad \sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x);$$

$$\tau_{xy} = G\cdot\gamma_{xy}$$

Relationships between E, ν, G, K

Związki pomiędzy E, ν, G, K

$$G = \frac{E}{2(1+\nu)} \qquad \text{as before}$$



- triaxial tension (trójosiowe rozciąganie)
- cuboid volume before loading (objętość prostopadłościanu przed obciążeniem)

$$V_0 = dx \cdot dy \cdot dz$$

- length of the sides of the cuboid after loading (długości boków po obciążeniu)

$$dx(1+\varepsilon_x),\ dy(1+\varepsilon_y),\ dz(1+\varepsilon_z)$$

- cuboid volume after loading (objętość prostopadłościanu po obciążeniu)

$$V = dx(1+\varepsilon_x) \cdot dy(1+\varepsilon_y) \cdot dz \cdot (1+\varepsilon_z)$$

- relative volume increase ϑ (względny przyrost objętości)

$$\vartheta = \frac{V - V_0}{V_0} = \frac{\Delta V}{V_0} = \frac{dx(1+\varepsilon_x) \cdot dy(1+\varepsilon_y) \cdot dz(1+\varepsilon_z) - dx \cdot dy \cdot dz}{dx \cdot dy \cdot dz}$$

$$\vartheta = (1+\varepsilon_x)\cdot(1+\varepsilon_y)\cdot(1+\varepsilon_z) - 1 = 1 + \varepsilon_x + \varepsilon_y + \varepsilon_z + \varepsilon_x\varepsilon_y + \varepsilon_y\varepsilon_z + \varepsilon_z \cdot \varepsilon_x + \varepsilon_x \cdot \varepsilon_y \cdot \varepsilon_z - 1 \cong \varepsilon_x + \varepsilon_y + \varepsilon_z$$

$$\boxed{\vartheta \cong \varepsilon_x + \varepsilon_y + \varepsilon_z}$$

$\varepsilon_x\varepsilon_y,\ \varepsilon_y\varepsilon_z,\ \varepsilon_z\varepsilon_x,\ \varepsilon_x\varepsilon_y\varepsilon_z$ (small)

$$\vartheta = \varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z) + \sigma_y - \nu(\sigma_z + \sigma_x) + \sigma_z - \nu(\sigma_x + \sigma_y)\right] = \frac{1}{E}\left[\sigma_x + \sigma_y + \sigma_z - 2\nu(\sigma_x + \sigma_y + \sigma_z)\right] =$$

$$\boxed{\vartheta = \frac{1-2\nu}{E}(\sigma_x + \sigma_y + \sigma_z)}$$

when the object is loaded with hydrostatic pressure
(gdy obiekt jest obciążony ciśnieniem hydrostatycznym)

$$\sigma_x = \sigma_y = \sigma_z = -p$$

$$\vartheta = \frac{1-2\nu}{E}(-p-p-p) = \frac{-3p(1-2\nu)}{E}$$

$$\vartheta = \frac{-p}{\frac{E}{3(1-2\nu)}} \Rightarrow K \qquad K = \frac{E}{3(1-2\nu)}$$

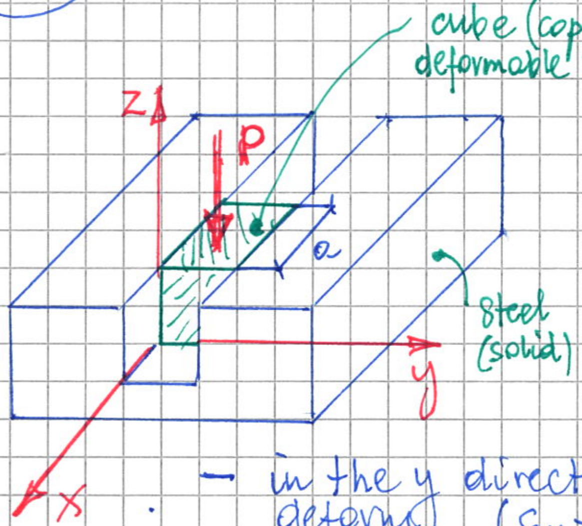
Bulk modulus
(Volume modulus, Modulus of compressibility)
moduł ściśliwości

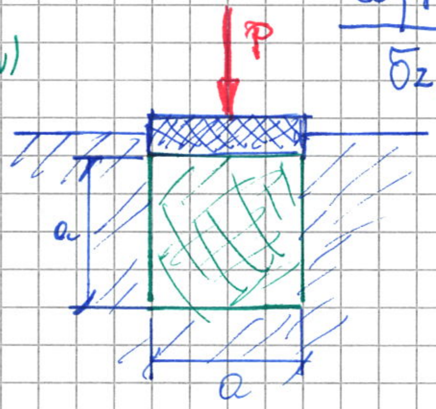
$$\boxed{\vartheta = \frac{-p}{K_0}}$$

$$\varepsilon = \frac{\sigma}{E} \; ; \quad \gamma = \frac{\tau}{G}$$

$E, G, K, \nu \rightarrow$ only 2 independent
(tylko 2 niezależne)

Ex. 1. / ćw. 1.





a, P, ν

$\sigma_z, \sigma_y - ?$

$\varepsilon_x - ?$

- in the y direction, the copper cube cannot deform ($\varepsilon_y = 0$)
(w kierunku y kostka Cu nie może się odkształcać)

$$\varepsilon_y = \frac{1}{E}\left(\sigma_y - \nu\sigma_z\right) = 0 \Rightarrow \sigma_y = \nu\sigma_z$$

$$\sigma_z = -\frac{P}{a^2}, \quad \sigma_y = \nu\cdot\sigma_z = \frac{-\nu\cdot P}{a^2}$$

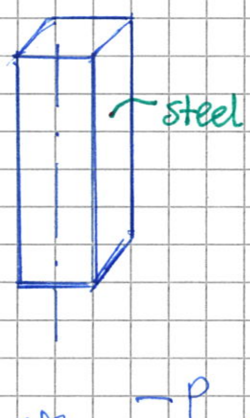
$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right] = \frac{1}{E}\left[0 - \nu\left(-\frac{\nu P}{a^2} - \frac{P}{a^2}\right)\right]$$

$$\varepsilon_x = \frac{1}{E}\left(\frac{\nu^2 P}{a^2} + \frac{\nu P}{a^2}\right) = \frac{\nu\cdot P}{a^2}\left(\nu + 1\right)$$

$P = 200$ kN, $\nu = 0{,}33$, $a = 40$ mm

$\sigma_z, \sigma_y, \varepsilon_x - ?$

Ex. 2. / ćw. 2



Hydrostatic pressure (ciśnienie hydrostatyczne) $p = 200$ MPa

$\nu = 0{,}3 \quad E = 2{,}1\cdot 10^5$ MPa

$\frac{\Delta V}{V} = \vartheta$ - relative volume increase - ?

$$K = \frac{E}{3(1-2\nu)} = \frac{2{,}1\cdot 10^5}{3(1-2\cdot 0{,}3)}$$

$$\vartheta = \frac{-p}{K} = \frac{-200\cdot 3\,(1-2\cdot 0{,}3)}{2{,}1\cdot 10^5} = -1{,}14\cdot 10^{-3} = -0{,}114\%$$