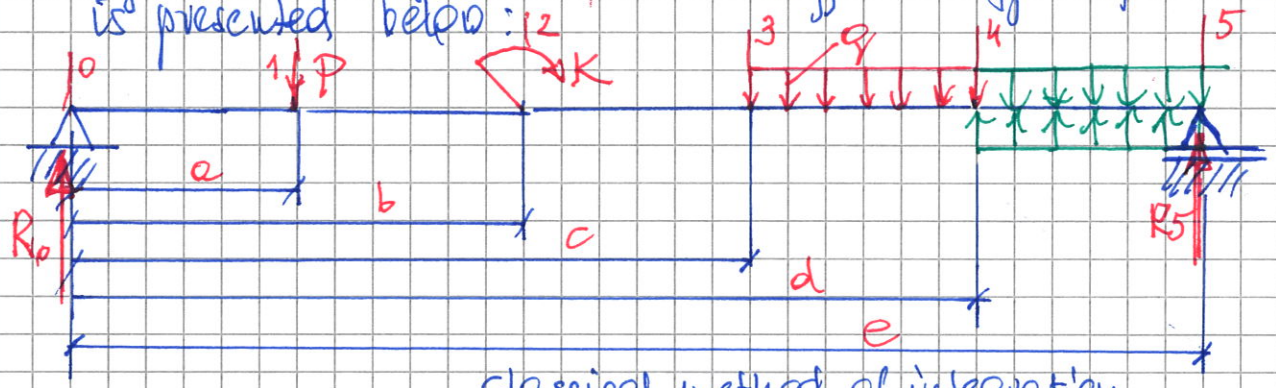
Bending 4

Initial Parameters Method
(Clebsch's method, Macaulay's method)

Metoda parametrów początkowych
(metoda Clebsch'a, metoda Macaulay'a)

Initial parameters method presents a modification of double integration method in case when a beam has several portions/sections and as result, expressions for bending moments are different for each portion/section. Initial parameter method allows us to obtain an equation of the classic curve of beam with any type of supports (rigid or elastic) and, most important, for any number of portions of a beam

A general case of beam under different types of loads is presented below:



Classical method of integration

$0 \le x_1 \le a$

$$M_1(x) = R_0 \cdot x$$

$$EI\frac{d^2y}{dx^2} = -R_0 \cdot x$$

$$EI\frac{dy}{dx} = -R_0 \frac{x^2}{2} + C_1$$

$$EIy = -R_0 \frac{x^3}{6} + C_1 x + D_1$$

same

$0 \le x_2 \le b$

$$M_2(x) = R_0 \cdot x - P(x-a) = R_0 \cdot x - P \cdot x + P \cdot a$$

$$EI\frac{d^2y}{dx^2} = -R_0 \cdot x + P \cdot x - P \cdot a$$

$$EI\frac{dy}{dx} = -R_0 \frac{x^2}{2} + P\frac{x^2}{2} - P \cdot a \cdot x + C_2$$

$$EIy = -R_0 \frac{x^3}{6} + P\frac{x^3}{6} - P a \frac{x^2}{2} + C_2 x + D_2$$

should be compared

Clebsch's method

$$M_1(x) = R_0 \cdot x$$

$$EI\frac{d^2y}{dx^2} = -R_0 x$$

$$EI\frac{dy}{dx} = -R_0 \frac{x^2}{2} + C_1$$

$$EIy = -R_0 \frac{x^3}{6} + C_1 x + D_1$$

$$M_2(x) = R_0 \cdot x - P(x-a)$$

$$EI\frac{d^2y}{dx^2} = -R_0 \cdot x + P(x-a)$$

$$EI\frac{dy}{dx} = -R_0 \frac{x^2}{2} + P\frac{(x-a)^2}{2} + C_2^{*}$$

$$EIy = -R_0 \frac{x^3}{6} + P\frac{(x-a)^3}{6} + C_2^{*} \cdot x + D_2^{*}$$

Comparison for the second section / period

assumption:

the same slopes and deflections regardless of the method of integration

slopes (the same)

classical — Clebsch's

$$-\cancel{R_0\frac{x^2}{2}}+P\frac{x^2}{2}-Pa\cdot x+C_2=-\cancel{R_0\frac{x^2}{2}}+P\frac{(x-a)^2}{2}+C_2^{*}$$

$$P\frac{x^2}{2}-P\cdot a\cdot x+C_2=\frac{P}{2}\left(x^2-2ax+a^2\right)+C_2^{*}$$

$$\cancel{P\frac{x^2}{2}}-\cancel{P\cdot a\cdot x}+C_2=\cancel{\frac{Px^2}{2}}-\cancel{P\cdot ax}+\frac{Pa^2}{2}+C_2^{*}$$

$$\boxed{C_2=C_2^{*}+\frac{Pa^2}{2}}\Rightarrow C_2^{*}=C_2-\frac{Pa^2}{2}$$

only constant

deflections (the same)

$$-\cancel{R_0\frac{x^3}{6}}+P\frac{x^3}{6}-Pa\frac{x^2}{2}+C_2\cdot x+D_2=$$

$$-\cancel{R_0\frac{x^3}{6}}+P\frac{(x-a)^3}{6}+C_2^{*}\cdot x+D_2^{*}$$

$$P\frac{x^3}{6}-Pa\frac{x^2}{2}+C_2\cdot x+D_2=\frac{P}{6}\left(x^3-3x^2a+3xa^2-a^3\right)$$

$$+C_2^{*}\cdot x+D_2^{*}$$

$$\cancel{P\frac{x^3}{6}}-\cancel{Pa\frac{x^2}{2}}+C_2\cdot x+D_2=\cancel{\frac{Px^3}{6}}-\cancel{\frac{Px^2a}{2}}+\frac{Pxa^2}{2}-\frac{Pa^3}{6}$$

$$+C_2^{*}\cdot x+D_2^{*}$$

$$\left(C_2^{*}+\frac{Pa^2}{2}\right)\cdot x+D_2=\frac{Pxa^2}{2}-\frac{Pa^3}{6}+C_2^{*}\cdot x+D_2^{*}$$

$$\cancel{C_2^{*}\cdot x}+\cancel{\frac{Pa^2x}{2}}+D_2=\cancel{\frac{Pxa^2}{2}}-\frac{Pa^3}{6}+\cancel{C_2^{*}\cdot x}+D_2^{*}$$

$$\boxed{D_2=D_2^{*}-\frac{Pa^3}{6}}\Rightarrow D_2^{*}=D_2+\frac{Pa^3}{6}$$

Fundamental differences between methods:

1. The equation of the bending moments should be written in such a way that the equation for the new portion section includes all the components of the equation for the previous section.

Równanie momentów gnących należy napisać tak, by w równaniu dla nowego przedziału wchodziły wszystkie składniki równania dla przedziałów poprzednich

$P_i(x-a_i)$ or $q_i\frac{(x-a_i)^2}{2}$

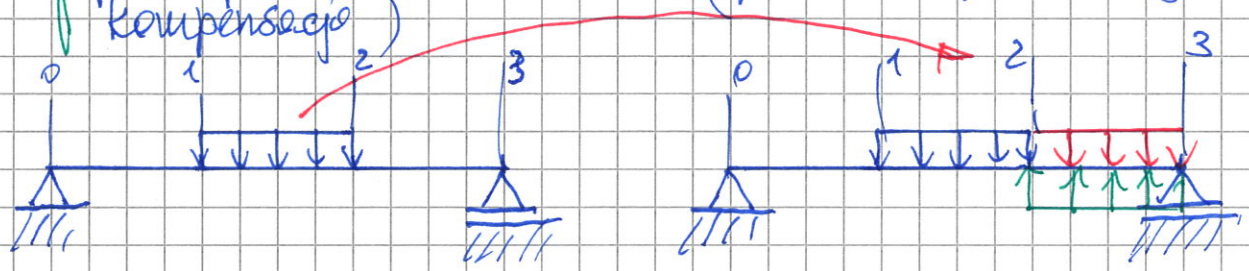
2. If a_i are the coordinates of the concentrated forces P_i or the beginning of the continous load q_i, then the expressions in parentheses should not be expanded. Integrate according to the scheme:

Jeżeli a_i są współrzędnymi obciążających sił skupionych P_i lub początków obciążenia ciągłego q_i, to w wyrażeniach typu $P_i(x-a_i)$ lub $q_i\frac{(x-a_i)^2}{2}$ nie należy rozwijać wyrażeń w nawiasach. Całkować według schematu

$$\int (x-a_i)^n dx = \frac{(x-a_i)^{n+1}}{n+1} + C$$

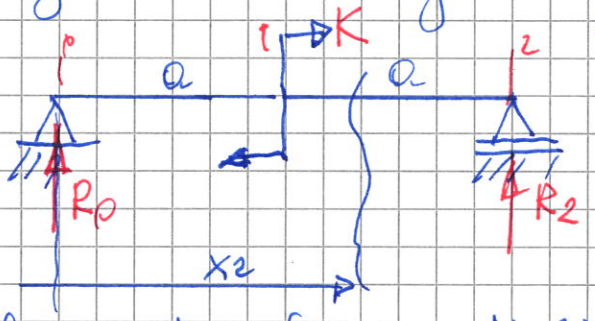
3. In the case of a continuous load, they should be presented in such a way that each started continous load runs to the end (if necessary, then with the use of compensation)

W przypadku obciążenia ciągłego należy tak je przedstawić, by każde zaczęte obciążenie ciągłe przebiegało aż do końca (jeśli trzeba, to z użyciem kompensacji)



(4) In the case of a concentrated moment, enter the coordinate of this moment into equation (with an exponent equal to zero).

W przypadku działania momentu skupionego należy do równania wprowadzić współrzędną tego momentu (z wykładnikiem równym zero).

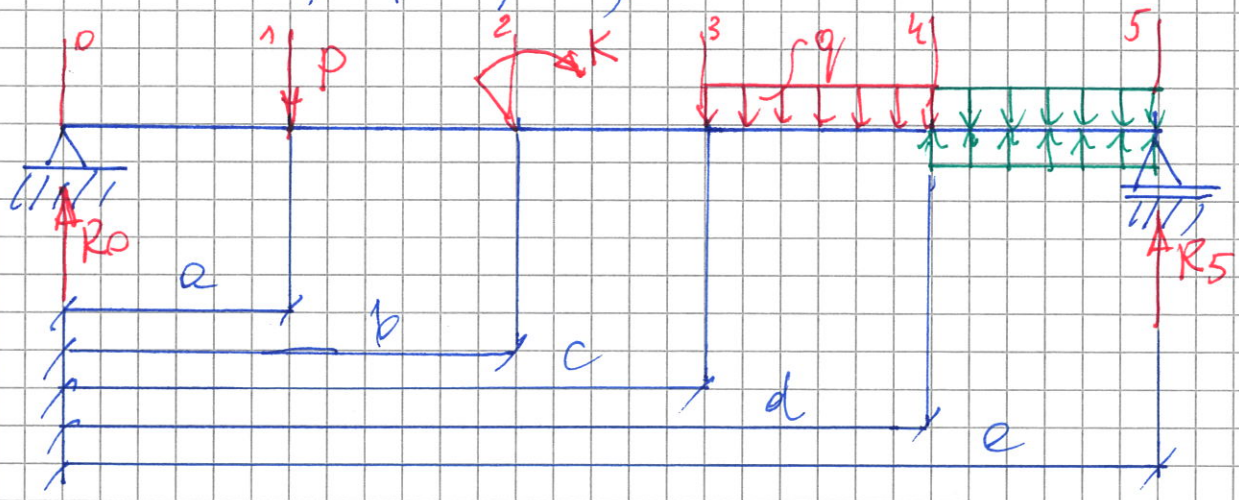


for second section: $M_2(x) = R_0 \cdot x + K(x-a)^0$

If we follow the above rules (1-4), the integer constants will be identical for all sections !

$C_1, C_2, C_3, \dots, C_i \Rightarrow C$

$D_1, D_2, D_3, \dots, D_i \Rightarrow D$



	0	1	2	3	4	5
$0 \le x_1 \le a,\ EI\frac{d^2y}{dx^2} =$	$-R_0 x$					
$a \le x_2 \le b,\ EI\frac{d^2y}{dx^2} =$	$-R_0 x$	$+P(x-a)$				
$b \le x_3 \le c,\ EI\frac{d^2y}{dx^2} =$	$-R_0 x$	$+P(x-a)$	$-K(x-b)^0$			
$c \le x_4 \le d,\ EI\frac{d^2y}{dx^2} =$	$-R_0 x$	$+P(x-a)$	$-K(x-b)^0$	$+q\frac{(x-c)^2}{2}$		
$d \le x_5 \le e,\ EI\frac{d^2y}{dx^2} =$	$-R_0 x$	$+P(x-a)$	$-K(x-b)^0$	$+q\frac{(x-c)^2}{2}$	$-q\frac{(x-d)^2}{2}$	

↑ Moments and diff. equations

↓ Slopes

	0	1	2	3	4	5
$EI\frac{dy}{dx} = C$	$-R_0\frac{x^2}{2}$					
$EI\frac{dy}{dx} = C$	$-R_0\frac{x^2}{2}$	$+P\frac{(x-a)^2}{2}$				
$EI\frac{dy}{dx} = C$	$-R_0\frac{x^2}{2}$	$+P\frac{(x-a)^2}{2}$	$-K(x-b)^1$			
$EI\frac{dy}{dx} = C$	$-R_0\frac{x^2}{2}$	$+P\frac{(x-a)^2}{2}$	$-K(x-b)$	$+q\frac{(x-c)^3}{6}$		
$EI\frac{dy}{dx} = C$	$-R_0\frac{x^2}{2}$	$+P\frac{(x-a)^2}{2}$	$-K(x-b)^0$	$+q\frac{(x-c)^3}{6}$	$-q\frac{(x-d)^3}{6}$	

↓ Deflections

$EIy = D + C\cdot x$	$-R_0\frac{x^3}{6}$					
$EIy = D + Cx$	$-R_0\frac{x^3}{6}$	$+P\frac{(x-a)^3}{6}$				
$EIy = D + C\cdot x$	$-R_0\frac{x^3}{6}$	$+P\frac{(x-a)^3}{6}$	$-K\frac{(x-b)^2}{2}$			
$EIy = D + Cx$	$-R_0\frac{x^3}{6}$	$+P\frac{(x-a)^3}{6}$	$-K\frac{(x-b)^2}{2}$	$+q\frac{(x-c)^4}{24}$		
$EIy = D + Cx$	$-R_0\frac{x^3}{6}$	$+P\frac{(x-a)^3}{6}$	$-K\frac{(x-b)^2}{2}$	$+q\frac{(x-c)^4}{24}$	$-q\frac{(x-d)^4}{24}$	

(*) (**)

$C, D - ?$

Boundary conditions

$x=0, y=0$ ⊛

$x=a, y=0$ ⊛⊛

It can therefore be said that in the Clebsch's method we write equations only for the last section

Thanks to the Clebsch's method we reduce the number of integration constants (so we need fewer boundary conditions)

for classical method

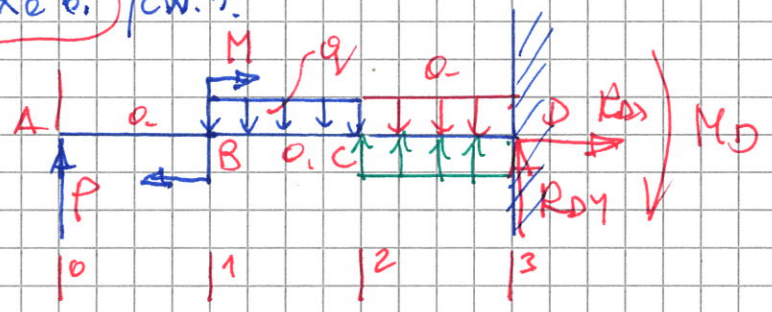
(5 sections: C_1, C_2, C_3, C_4, C_5
D_1, D_2, D_3, D_4, D_5 | 10 constants (10 boundary conditions)

for Clebsch's method

5 sections but: C, D | 2 constants (2 b. conditions)

Attention to hinged beams, special case (will be explained separately) !

Ex. 6. / cw. 1.



$q, a, P = qa, M = qa^2$
$EI = const$

$y_A, v_C - ?$
(Clebsch's method)

I static eqs.

① $\sum P_{ix} = 0 \Rightarrow R_{Dx} = 0$

② $\sum P_{iy} = P - qa + R_{Dy} = 0$

③ $\sum M_i^D = P \cdot 3a + M - qa \cdot \frac{3}{2}a + M_D = 0$

$\Rightarrow R_{Dx}, R_{Dy}, M_D$

$$M(x) = \begin{array}{c|c|c|c|} 0 & 1 & 2 & 3 \\ P\cdot x & +M(x-a)^0 - q\frac{(x-a)^2}{2} & +q\frac{(x-2a)^2}{2} & \end{array}$$

$$EI\frac{d^2y}{dx^2} = \begin{array}{c|c|c|c|} & -P\cdot x & -M(x-a)^0 + q\frac{(x-a)^2}{2} & -q\frac{(x-2a)^2}{2} \end{array}$$

$$EI\frac{dy}{dx} = C + \begin{array}{c|c|c|c|} & -P\frac{x^2}{2} & -M(x-a) + q\frac{(x-a)^3}{6} & -q\frac{(x-2a)^3}{6} \end{array}$$

$$EIy = Cx + D \begin{array}{c|c|c|c|} & -P\frac{x^3}{6} & -M\frac{(x-a)^2}{2} + q\frac{(x-a)^4}{24} & -q\frac{(x-2a)^4}{24} \end{array}$$

4 (*)

(*)

(**)

3* C, D – ?

Boundary conditions

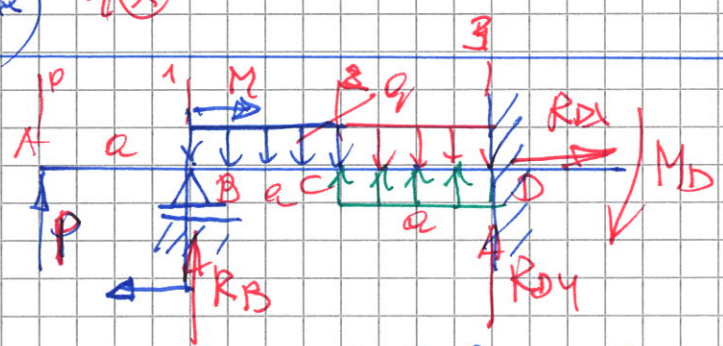
$x = 3a$, $v = 0$ (*)

$x = 3a$, $y = 0$ (**)

$y_A = y(x=0)$ 3*

$v_C = v(x=2a)$ 4 (*)

Ex. 2. / ćw. 4.



I static eqs.

1. $\sum P_{ix} = R_{Dx} = 0$
2. $\sum P_{iy} = P + R_B - qa + R_{Dy} = 0$
3. $\sum M_i^D = P\cdot 3a + R_B\cdot 2a - qa\frac{3}{2}a + M + M_D = 0$

4 - 3 => 1x hyperstatic beam

$q, a, P = qa \quad M = qa^2$

$EI = const$

reactions – ?

R_B, R_{Dx}, R_{Dy}, M_D

⇓

4 reactions

(+ C, D)

	0	1	2	3
$M(x) =$	$P \cdot x$	$+R_B(x-a) + M(x-a)^0 - q\frac{(x-a)^2}{2}$	$+ q\frac{(x-2a)^2}{2}$	
$EI\frac{d^2y}{dx^2} =$	$-P \cdot x$	$-R_B(x-a) - M(x-a)^0 + q\frac{(x-a)^2}{2}$	$- q\frac{(x-2a)^2}{2}$	
$EI\frac{dy}{dx} = C +$	$-P\frac{x^2}{2}$	$-R_B\frac{(x-a)^2}{2} - M(x-a) + q\frac{(x-a)^3}{6}$	$- q\frac{(x-2a)^3}{6}$	**
$EIy = C_1 x + D$	$-P\frac{x^3}{6}$	$-R_B\frac{(x-a)^3}{6} - M\frac{(x-a)^2}{2} + q\frac{(x-a)^4}{24}$	$- q\frac{(x-2a)^4}{24}$	
	*		3*	

Boundary conditions

1. $x = a$, $y = 0$ *
2. $x = 3a$, $\vartheta = 0$ **
3. $x = 3a$, $y = 0$ 3*

3 static eqs + 3 b. conditions = 4 reactions + 2 constants

$6 = 6$

$4 = 4$