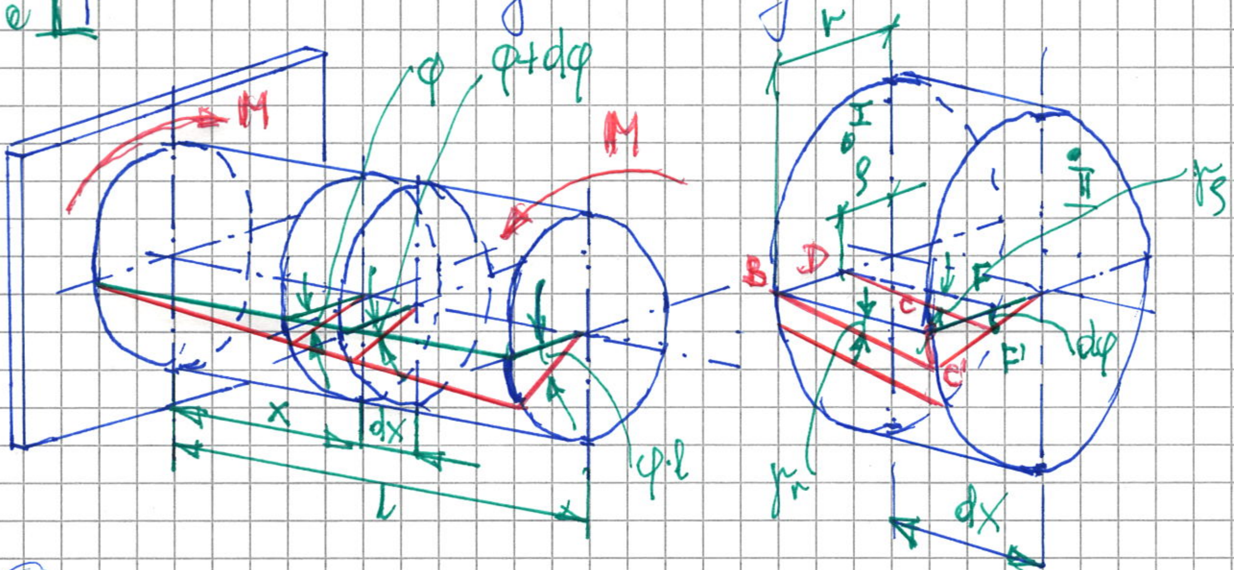
Torsion. Torsion of circular shafts

Skręcanie. Skręcanie wałów okrągłych

[Circular shaft are commonly used for to transmit power in rotating machinery.



A. Simplifying assumptions (założenia)

- During the deformation, the cross section are not distored in any manner - they remain plane, and the radius r does not change. In addition, the length l of the shaft remainst constant.
- Based on this observations, we make the following assumptions
  - circular cross section remain plane (do not warp) and perpendicular to the axis of the shaft,
  - cross section do not deform (there is no strain in the plane of the cross section),
  - the distances between cross sections do not change (the axis normal strain is zero),
- Each cross section rotates as a rigid entity about the axis of the shaft.

B. Compatibility (zgodność)

- Because the cross sections are separated by an infinitesimal distance dx the differences in the rotations, denoted by the $d\varphi$ angle, is also infinitesimal
- As the cross sections underge the relative rotation $d\varphi$, BC line deforms into helix BC'. Helix angle γ is the shear strain of the element

From the geometry of figure, we obtain;

for $\rho = r$ $\qquad \gamma_r = \dfrac{CC'}{BC}$ $\qquad$ but $\quad BC = DF = dx$

for ρ $\qquad \gamma = \dfrac{FF'}{DF}$

$$\underbrace{FF' = dx \cdot \gamma \quad \text{and} \quad FF' = \rho \cdot d\varphi}$$

$$\boxed{\gamma = \frac{d\varphi}{dx} \cdot \rho}$$

The corresponding shear stress is determined from Hooke's law

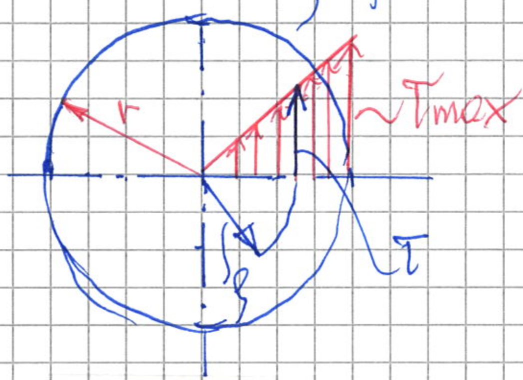
$$\boxed{\gamma = \frac{\tau}{G}}$$

$$\boxed{\tau = \gamma \cdot G = G \cdot \underbrace{\frac{d\varphi}{dx}}_{const} \cdot \rho}$$

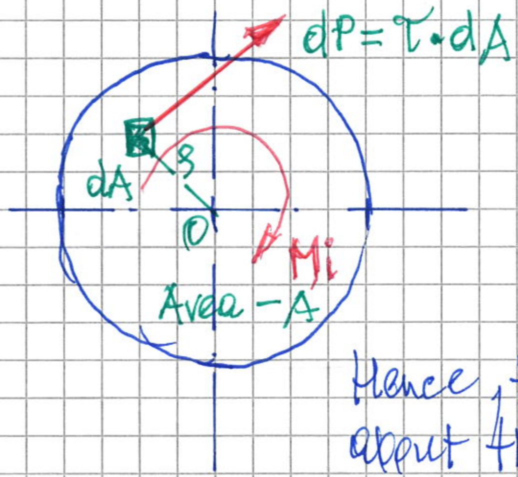
for selected / defined cross section $\dfrac{d\varphi}{dx} = const$

$\dfrac{d\varphi}{dx}$ — is the angle of twist per unit ~~leg~~ length

The shear stress varies linearly with the radial distance ρ from the axial of the shaft.



③ Equilibrium (równowaga)



The shear force acting on elementary area dA is

$$dP = \tau \cdot dA = G \cdot \frac{d\varphi}{dx} \cdot \rho \cdot dA$$

directed perpendicular to the radius.

Hence, the moment (torque) of the dP about the center O is

$$\rho \cdot dP = G \cdot \frac{d\varphi}{dx} \cdot \rho^2 \cdot dA$$

Summing the contributions and equating the result to the internal torque yields Mi

$$\int_A \rho \, dP = M_i, \text{ or}$$

$$\int_A \underbrace{G}_{\text{const}} \cdot \underbrace{\frac{d\varphi}{dx}}_{\text{const.}} \cdot \rho^2 \cdot dA = M_i$$

$$G \cdot \frac{d\varphi}{dx} \int_A \rho^2 \cdot dA = M_i \quad \text{but} \quad \int_A \rho^2 dA = I_o$$

I_o - polar moment of inertia of the cross-sectional area

$$G \cdot \frac{d\varphi}{dx} \cdot I_o = M_i = M \quad \left(\text{but } M_i = M \right.$$ (external torque moment)

$$\boxed{\frac{d\varphi}{dx} = \frac{M}{G \cdot I_o}}$$

$G I_o$ – torsional (torque) stiffness

$$\boxed{\varphi = \int_0^l \frac{M}{G \cdot I_o} dx \Rightarrow \frac{M \cdot l}{G \cdot I_o}}$$

the angle of twist

but $\frac{d\varphi}{dx} = \frac{\tau}{G \cdot \rho}$, and $\frac{d\varphi}{dx} = \frac{M}{G I_o}$

$$\tau = \frac{d\varphi}{dx} \cdot G \cdot \rho = \frac{M}{\cancel{G} \cdot I_o} \cdot \cancel{G} \cdot \rho = \frac{M \cdot \rho}{I_o}$$

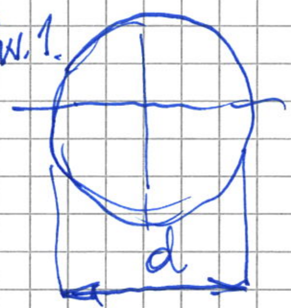
$$\boxed{\tau = \frac{M}{I_o} \cdot \rho}$$ but τ_{max} for $\rho = r$

$$\boxed{\tau_{max} = \frac{M}{I_o} \cdot r}$$ $\frac{I_o}{r} = W_o$

W_o - polar modulus (section modulus for torsion) wskaźnik wytrzym. na skręcanie

D) Calculations, examples

Ex. 1. rys. 1.



$M, \tau_{max}, G, l, \varphi_{max}$

$d - ?$

$$I_o = \frac{\pi d^4}{32} = \frac{\pi r^4}{2}$$

$$W_o = \frac{\pi d^3}{16}$$

$$\tau_{max} \leq \frac{M}{\frac{\pi d^3}{16}}$$

$$\tau_{max} \leq \frac{M}{I_o} r$$

$$\tau_{max} \leq \frac{M}{W_o}$$

$$\tau_{max} = \frac{16 M}{\pi d^3}$$

$$\boxed{d \geq \sqrt[3]{\frac{16 M}{\pi \tau_{max}}}}$$

$$\varphi_{max} \leq \frac{M \cdot l}{G \cdot I_o}$$ $$\varphi_{max} = \frac{M \cdot l \cdot 32}{G \cdot \pi d^4}$$

$$d \geq \sqrt[4]{\frac{32 M \cdot l}{\pi \cdot G \cdot \varphi_{max}}} \quad \varphi \text{ (radians)}$$

$$d \geq \sqrt[4]{\frac{180 \cdot 32 \cdot M \cdot l}{\pi \cdot \pi \cdot G \cdot \varphi_{max}}} \quad \varphi \text{ (degrees)}$$

$$d_{(\varphi_{max})} > d_{(\tau_{max})}$$

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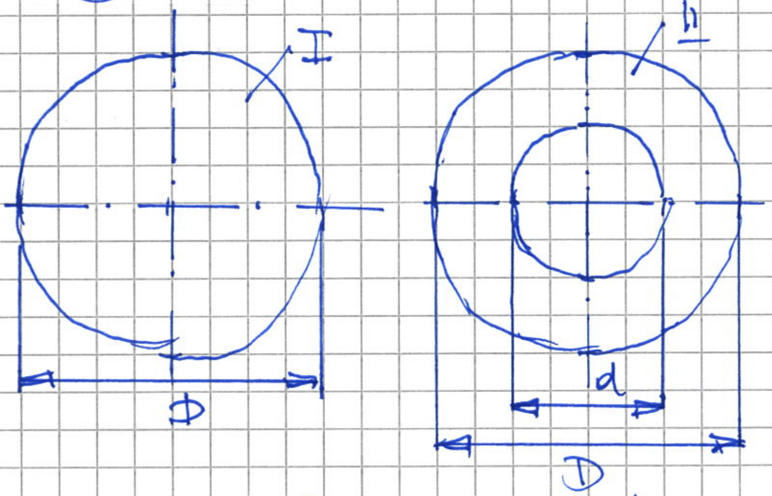
Ex. 2 / ćw. 2



D, $d = 0{,}5 D$, γ, l

$d = \alpha \cdot D$ $\quad \alpha = 0{,}5$

W - weight

τ - stress

$$\frac{\tau_{II}}{\tau_I} = ? \qquad \frac{W_{II}}{W_I} = ?$$

$$W_I = \frac{\pi D^2}{4} \cdot \gamma \cdot l$$

$$W_{II} = \frac{\pi}{4}\left(D^2 - d^2\right) \cdot \gamma \cdot l$$

$$\cancel{\frac{W_{II}}{W_I} = \frac{\pi D^2}{4} \cdot \gamma \cdot l \cdot \frac{4}{\pi\left(D^2 - d^2\right) \gamma \cdot l}}$$

$$\frac{W_{II}}{W_I} = \frac{\pi\left(D^2 - d^2\right) \gamma \cdot l}{4} \cdot \frac{4}{\pi D^2 \cdot \gamma \cdot l}$$

$$\cancel{\frac{W_{II}}{W_I} = \frac{D^2}{D^2 - d}}$$

$$\frac{W_{II}}{W_I} = \frac{D^2 - d^2}{D^2} = \frac{D^2 - (0{,}5D)^2}{D^2} =$$

$$= \frac{D^2 - 0{,}25 D^2}{D^2} = 1 - 0{,}25 = 0{,}75 \quad (75\%)$$

$$W_{oI} = \frac{\pi D^3}{16}$$

$$W_{oII} = \frac{\pi}{16}\left(D^3 - d^3\right)$$

$$\tau_I = \frac{M}{W_{oI}} \qquad \tau_{II} = \frac{M}{W_{oII}}$$

$$\frac{\tau_{II}}{\tau_I} = \frac{D^3 - d^3}{D^3} = \frac{D^3 - (0{,}5D)^3}{D^3} =$$

$$= 1 - 0{,}125 = 0{,}875 \quad (87{,}5\%)$$

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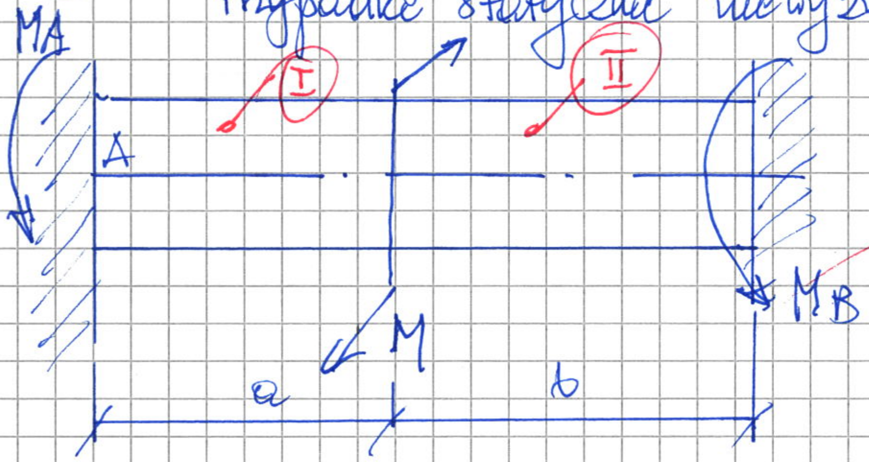
E Statically indeterminate problems (hyperstatic)

Przypadki statycznie niewyznaczalne



M, a, b, d, G

τ_I, τ_{II} - ?

$I_0 = \frac{\pi d^4}{32}$

$W_0 = \frac{\pi d^3}{16}$

I static eqs.

(1) $\sum M_i = -M_A + M - M_B = 0$

2 reactions - 1 st. eq $\Rightarrow$ 1× hyperstatic

II Geometrical eq.

$$\varphi_{I} + \varphi_{II} = \varphi = 0$$

III Physical eqs.

$$\varphi_I = \frac{-M_A \cdot a}{GI_0} \; ; \quad \varphi_{II} = \frac{(-M_A + M) \cdot b}{GI_0} \Bigg| \cdot \cancel{GI_0}$$

$$-\frac{M_A \cdot a}{GI_0} + \frac{(M - M_A) \cdot b}{GI_0} = 0 \Bigg| \cdot GI_0$$

$$-M_A \cdot a + M \cdot b - M_A \cdot b = 0$$

$$-M_A(a+b) + M \cdot b = 0 \Rightarrow M_A = M\frac{b}{a+b}$$

$$\tau_I = \frac{M_A}{\frac{\pi d^3}{16}} = \frac{16 M_A}{\pi d^3} = \frac{16\, Mb}{\pi d^3 (a+b)}$$

$$\tau_{II} = \frac{M - M_A}{\frac{\pi d^3}{16}} = \frac{16\,(M - M_A)}{\pi d^3} = \frac{16}{\pi d^3}\left(M - M\frac{b}{a+b}\right) = \frac{16 M a}{\pi d^3 (a+b)}$$