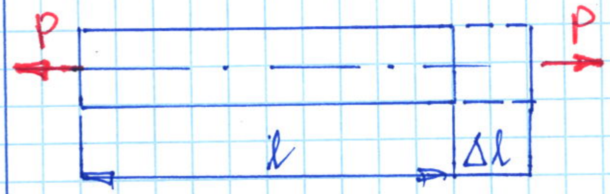
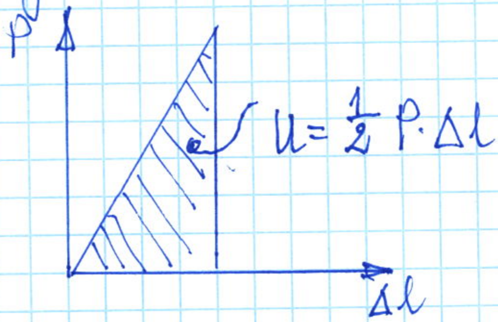
Internal Strain Energy. Castigliano's theorem

When an external force acts upon an elastic body and deforms it, the work done by the force is stored within the body in the form of strain energy. For a straight bar subject to the tensile force P, the internal strain energy U is given by:





$$U=\frac{1}{2}P\cdot\Delta l \quad ; \quad \Delta l=\frac{P\cdot l}{EA} \quad ; \quad U=\frac{1}{2}P\cdot\frac{P\cdot l}{E\cdot A}$$

$$\boxed{U=\frac{1}{2}\frac{P^2\cdot l}{EA}} \quad , \quad \boxed{\frac{dU}{dx}=\frac{1}{2}\frac{P^2}{EA}}$$

elementary energy

For a circular bar of length l subject to a torque M_s, the internal strain energy U is given by

$$\boxed{U=\frac{1}{2}\frac{M_s^2\, l}{GI_0}} \quad , \quad \boxed{\frac{dU}{dx}=\frac{1}{2}\frac{M_s^2}{GI_0}}$$

elementary energy

For a bar of length l subject to a bending moment M_g, the internal strain energy U is given by

$$\boxed{U=\frac{1}{2}\frac{M_g^2\cdot l}{EI}} \quad ; \quad \boxed{\frac{dU}{dx}=\frac{1}{2}\frac{M_g^2}{EI}}$$

elementary energy

For shear force T, respectively

$$\boxed{U=\frac{1}{2}\beta\frac{T^2\cdot l}{G\cdot A}} \quad ; \quad \boxed{\frac{dU}{dx}=\frac{1}{2}\beta\frac{T^2}{G\cdot A}}$$

elementary energy

Note that in each of these expressions the external load always occurs in the form of a squared magnitude, hence each of these energy expressions is always a positive scalar quantity.

Castigliano's theorem

This theorem is extremely useful for finding displacements of elastic bodies subject to axial loads, torsion, bending, shear force or any combination of these loadings. The theorem states that the partial derivative of the total internal strain energy with respect to any external applied force yields the displacement under the point of application of that force in the direction of that force. Here the terms force P_i and displacement p_i are used in their generalized sense and could either indicate a usual force and its linear displacement, or a couple and the corresponding angular displacement

$$\boxed{p_i = \frac{\partial U}{\partial P_i}}$$

P_i - generalised force

p_i - generalised displacement

or, second version

$$\boxed{P_i = \frac{\partial U}{\partial p_i}}$$

for hyperstatic object, where

$P_i = R_i$ - hyperstatic generalised force

$$\boxed{\frac{\partial U}{\partial R_i} = p_i = 0} \quad !$$

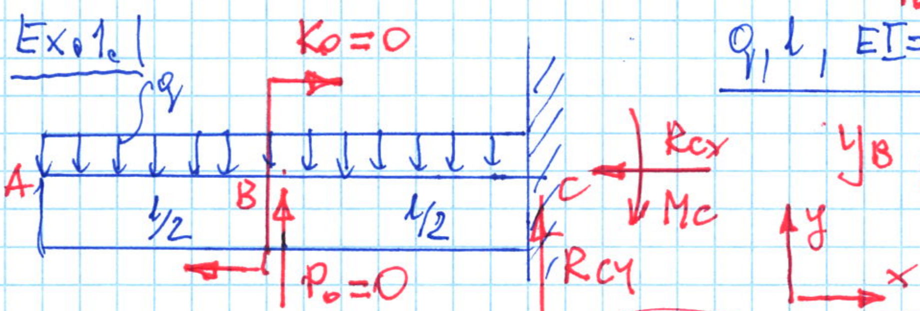
Menabrea - Castigliano theorem

P_0, K_0 - fictitious loads

Ex. 1.

$q, l, EI = const, \; P_0 = 0, \; K_0 = 0$

$y_B, \vartheta_B - ?$



(I) Static equations (without P_0, K_0)

(1) $\Sigma P_{ix} = -R_{cx} = 0$

(2) $\Sigma P_{iy} = -q \cdot l + R_{cy} = 0 \Rightarrow R_{cy} = q l$

(3) $\Sigma M_i^{c} = -q l \frac{l}{2} + M_c = 0$

$M_c = \frac{q l^2}{2}$

$\underline{0 \le x_1 \le \frac{l}{2}}$

$$M_1(x) = -\frac{qx^2}{2} \Rightarrow \frac{\partial M_1(x)}{\partial P_0} = 0 \qquad \frac{\partial M_1(x)}{\partial K_0} = 0$$

$\underline{\frac{l}{2} \le x_2 \le l}$

$$M_2(x) = -\frac{qx^2}{2} + P_0\left(x - \frac{l}{2}\right) + K_0 \Rightarrow \frac{\partial M_2(x)}{\partial P_0} = x - \frac{l}{2}$$

$$\frac{\partial M_2(x)}{\partial K_0} = 1$$

Displacements (for B)

$$y_B = \frac{\partial U}{\partial P_0} \qquad \vartheta_B = \frac{\partial U}{\partial K_0}$$

$$U = \int_0^{\frac{l}{2}} \frac{M_1^2(x)}{2EI}dx + \int_{\frac{l}{2}}^{l} \frac{M_2^2(x)}{2EI}dx = \frac{1}{2EI}\left[\int_0^{\frac{l}{2}} M_1^2(x)dx + \int_{\frac{l}{2}}^{l} M_2^2(x)dx\right]$$

$$y_B = \frac{\partial U}{\partial P_0} = \frac{1}{2EI}\left[2\int_0^{\frac{l}{2}} M_1(x)\cdot\frac{\partial M_1(x)}{\partial P_0}dx + 2\int_{\frac{l}{2}}^{l} M_2(x)\cdot\frac{\partial M_2(x)}{\partial P_0}dx\right] =$$

$$\boxed{y_B = \frac{1}{EI}\left[\int_0^{\frac{l}{2}} M_1(x)\cdot\frac{\partial M_1(x)}{\partial P_0}dx + \int_{\frac{l}{2}}^{l} M_2(x)\cdot\frac{\partial M_2(x)}{\partial P_0}dx\right] = ?}$$

$$\boxed{\vartheta_B = \frac{1}{EI}\left[\int_0^{\frac{l}{2}} M_1(x)\cdot\frac{\partial M_1(x)}{\partial K_0}dx + \int_{\frac{l}{2}}^{l} M_2(x)\cdot\frac{\partial M_2(x)}{\partial K_0}dx\right] = ?}$$

$$y_B = \frac{1}{EI}\left[\overbrace{\int_0^{\frac{l}{2}}\left(-\frac{qx^2}{2}\right)\cdot 0 \cdot dx}^{0} + \int_{\frac{l}{2}}^{l}\left[-\frac{qx^2}{2} + \overset{=0}{P_0}\left(x-\frac{l}{2}\right) + \overset{=0}{K_0}\right]\cdot\left(x-\frac{l}{2}\right)dx\right] =$$

$$= \frac{1}{EI}\int_{\frac{l}{2}}^{l}\left(-\frac{qx^2}{2}\right)\left(x-\frac{l}{2}\right)dx = \frac{1}{EI}\int_{\frac{l}{2}}^{l}\left(-\frac{qx^3}{2} + \frac{qx^2 l}{4}\right)dx =$$

$$\frac{1}{EI}\left(-\frac{qx^4}{8} + \frac{qx^3 l}{12}\right)\Bigg|_{\frac{l}{2}}^{l} = \frac{1}{EI}\left(-\frac{ql^4}{8} + \frac{ql^4}{12} + \frac{q}{8}\frac{l^4}{16} - \frac{ql}{12}\frac{l^3}{8}\right)$$

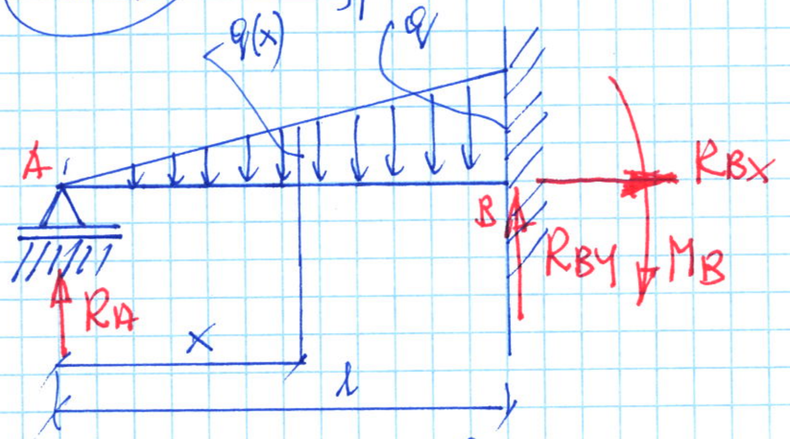
$$= \frac{ql^4}{EI}\left(\frac{-48+32+3-4}{384}\right) = -\frac{17}{384}\frac{ql^4}{EI}$$

$$v_B = \frac{1}{EI}\left[\int_0^{\frac{l}{2}} -\frac{qx^2}{2}\cdot 0\cdot dx + \int_{\frac{l}{2}}^{l}\left[-\frac{qx^2}{2} + \overset{=0}{P_0}\left(x-\frac{l}{2}\right) + \overset{=0}{K_0}\right]\cdot 1\, dx\right] =$$

$$= \frac{1}{EI}\int_{\frac{l}{2}}^{l} -\frac{qx^2}{2}\cdot 1\cdot dx = \frac{1}{EI}\int_{\frac{l}{2}}^{l} -\frac{qx^2}{2}\, dx = \frac{1}{EI}\left(-\frac{qx^3}{6}\right)\Bigg|_{\frac{l}{2}}^{l} =$$

$$= \frac{1}{EI}\left(-\frac{ql^3}{6} + \frac{q}{6}\frac{l^3}{8}\right) = \frac{1}{EI}\left(-\frac{ql^3}{6} + \frac{ql^3}{48}\right) = \underline{-\frac{7}{48}\frac{ql^3}{EI}}$$

Ex. 2. hyperstatic beam



$q,\ l,\ EI = \text{const}$

reactions – ?

$R_A,\ R_{Bx},\ R_{BY},\ M_B$ – ?

I Static equations

1) $\sum P_{ix} = R_{Bx} = 0$

2) $\sum P_{iy} = R_A - \frac{1}{2}ql + R_{BY} = 0$

3) $\sum M_i^B = R_A\cdot l - \frac{1}{2}ql\cdot\frac{l}{3} + M_B = 0$

$R_A\cdot l - \frac{ql^2}{6} + M_B = 0$

4 reactions – 3 static equations ⇒ 1x hyperstatic beam

we selected R_A as hyperstatic reaction

II Add. geometrical equation / Menabre'a – Castigliano theorem

4) $\underline{y_A = 0} \Rightarrow \underline{y_A = \frac{\partial U}{\partial R_A} = 0}$!

4) $y_A = \frac{1}{EI}\int_0^l M(x)\cdot\frac{\partial M(x)}{\partial R_A}\, dx = 0$

$$M(x) = R_A \cdot x - \frac{qx^3}{6l}, \qquad \frac{\partial M(x)}{\partial R_A} = x$$

4 $$y_A = \frac{1}{EI}\int_0^l \left(R_A \cdot x - \frac{qx^3}{6l}\right)\cdot x\,dx = \frac{1}{EI}\int_0^l \left(R_A x^2 - \frac{qx^4}{6l}\right)dx =$$

$$= \frac{1}{EI}\left(R_A \frac{x^3}{3} - \frac{qx^5}{30l}\right)\Big|_0^l = \frac{1}{EI}\left(R_A \frac{l^3}{3} - \frac{ql^5}{30l}\right) =$$

$$= \frac{1}{EI}\left(R_A \frac{l^3}{3} - \frac{ql^4}{30}\right) = 0 \Rightarrow R_A \frac{l^3}{3} = \frac{ql^4}{30}$$

$$R_A = \frac{ql}{10} \Rightarrow \left.\begin{matrix} 1 \\ 2 \\ 3 \end{matrix}\right\} \text{static equation}$$

from 2 $R_{By} = \cancel{ql}\ \frac{ql}{2} - R_A = \frac{ql}{2} - \frac{ql}{10} = \frac{4ql}{10} = \frac{2ql}{5}$

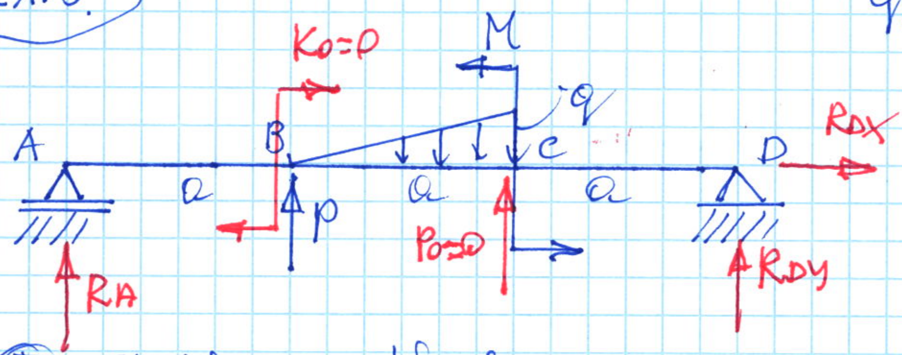
from 3 $M_B = \frac{ql^2}{6} - R_A \cdot l = \frac{ql^2}{6} - \frac{ql^2}{10} = \frac{2ql^2}{30} = \frac{ql^2}{15}$

$$R_A = \frac{ql}{10}, \quad R_{By} = \frac{2ql}{5}, \quad M_B = \frac{ql^2}{15}, \quad R_{Bx} = 0$$

Ex. 3.

$q, a, P = qa, M = qa^2$

$EI = const$



$y_B, \vartheta_B, y_C, \vartheta_C$ − ?

I static equations

1 $\Sigma P_{ix} = R_{Dx} = 0$

2 $\Sigma P_{iy} = R_A + P - \frac{1}{2}qa + P_0 + R_{Dy} = 0$

3 $\Sigma M_i^C = R_A \cdot 3a + K_0 + P \cdot 2a - \frac{1}{2}qa \cdot \frac{4}{3}a - M + P_0 \cdot a = 0$

$$R_A = \frac{1}{3a}\left(-K_0 - 2Pa + \frac{2}{3}qa^2 + M - P_0 \cdot a\right)$$

$$R_A = \frac{-K_0}{3a} - \frac{2}{3}P + \frac{2}{9}qa + \frac{M}{3a} - \frac{P_0}{3}$$

$$R_A = f(q, P, M, P_0, K_0) \ !$$

$$y_B = \frac{\partial U}{\partial P}\ ;\quad \vartheta_B = \frac{\partial U}{\partial K_0}\ ;\quad y_C = \frac{\partial U}{\partial P_0}\ ;\quad \vartheta_C = \frac{\partial U}{\partial M}$$

$0 \le x_1 \le a$

$$M_1(x) = R_A \cdot x = \left(-\frac{K_0}{3a} - \frac{2}{3}P + \frac{2}{9}qa + \frac{M}{3a} - \frac{P_0}{3}\right)\cdot x$$

$$\frac{\partial M_1(x)}{\partial P} = -\frac{2}{3}\cdot x\ ;\quad \frac{\partial M_1(x)}{\partial K_0} = -\frac{x}{3a}\ ;\quad \frac{\partial M_1(x)}{\partial P_0} = -\frac{x}{3}$$

$$\frac{\partial M_1(x)}{\partial M} = \frac{x}{3a}$$

$a \le x_2 \le 2a$

$$M_2(x) = R_A \cdot x + K_0 + \cancel{P}\,\cancel{x}^{\,P(x-a)} - q\frac{(x-a)^3}{6a} =$$

$$= \left(-\frac{K_0}{3a} - \frac{2}{3}P + \frac{2}{9}qa + \frac{M}{3a} - \frac{P_0}{3}\right)\cdot x + K_0 + \cancel{Px}^{\,P(x-a)} - q\frac{(x-a)^3}{6a}$$

$$\frac{\partial M_2(x)}{\partial P} = -\frac{2}{3}\cancel{x} + x - a = \frac{x}{3} - a\ ;\quad \frac{\partial M_2(x)}{\partial K_0} = -\frac{x}{3a} + 1 = 1 - \frac{x}{3a}$$

$$\frac{\partial M_2(x)}{\partial P_0} = -\frac{x}{3}\ ;\quad \frac{\partial M_2(x)}{\partial M} = \frac{x}{3a}$$

$2a \le x_3 \le 3a$

$$M_3(x) = R_A \cdot x + K_0 + P\cdot(x-a) - \frac{1}{2}qa\left(x - \frac{5}{3}a\right) + P_0(x-2a)$$

$$= \left(-\frac{K_0}{3a} - \frac{2}{3}P + \frac{2}{9}qa + \frac{M}{3a} - \frac{P_0}{3}\right)\cdot x + K_0 + P(x-a) - \frac{1}{2}qa\left(x - \frac{5}{3}a\right) + P_0(x-2a) - M$$

$$\frac{\partial M_3(x)}{\partial P} = -\frac{2}{3}x + x - a = \frac{x}{3} - a\ ;\quad \frac{\partial M_3(x)}{\partial K_0} = -\frac{x}{3a} + 1 = 1 - \frac{x}{3a}$$

$$\frac{\partial M_3(x)}{\partial P_0} = -\frac{x}{3} + x - 2a = \frac{2}{3}x - 2a\ ;\quad \frac{\partial M_3(x)}{\partial M} = \frac{x}{3a} - 1$$

$$y_B = \frac{1}{EI}\left[\int_0^a M_1(x)\cdot\frac{\partial M_1(x)}{\partial P}dx + \int_a^{2a} M_2(x)\frac{\partial M_2(x)}{\partial P}dx + \int_{2a}^{3a} M_3(x)\cdot\frac{\partial M_3(x)}{\partial P}dx\right]$$

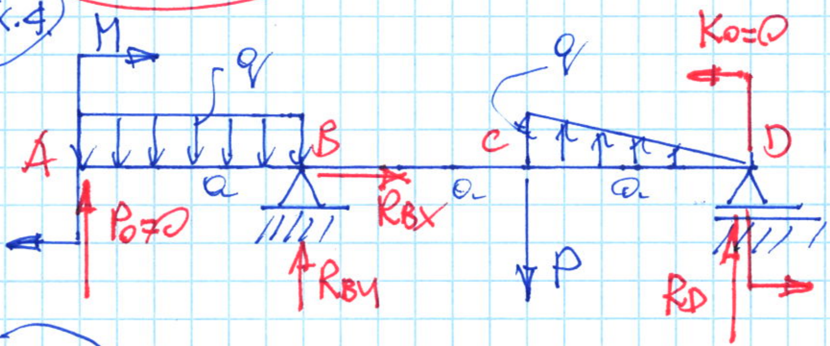
$$\vartheta_B = \frac{1}{EI}\left[\int_0^a M_1(x)\frac{\partial M_1(x)}{\partial K_0}dx + \int_a^{2a} M_2(x)\cdot\frac{\partial M_2(x)}{\partial K_0}dx + \int_{2a}^{3a} M_3(x)\cdot\frac{\partial M_3(x)}{\partial K_0}dx\right]$$

$$y_C = \frac{1}{EI}\left[\int_0^a M_1(x)\cdot\frac{\partial M_1(x)}{\partial P_0}dx + \int_a^{2a} M_2(x)\cdot\frac{\partial M_2(x)}{\partial P_0}dx + \int_{2a}^{3a} M_3(x)\cdot\frac{\partial M_3(x)}{\partial P_0}dx\right]$$

$$\vartheta_C = \frac{1}{EI}\left[\int_0^a M_1(x)\cdot\frac{\partial M_1(x)}{\partial M}dx + \int_a^{2a} M_2(x)\cdot\frac{\partial M_2(x)}{\partial M}dx + \int_{2a}^{3a} M_3(x)\cdot\frac{\partial M_3(x)}{\partial M}dx\right]$$

Homework

Ex. 4

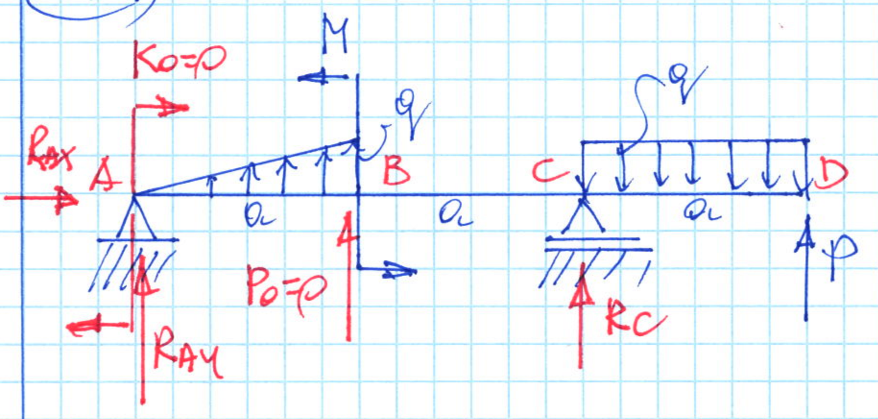


$q, a, P=qa, M=qa^2$

$EI = const$

y_A, y_C, ϑ_D - ?

Ex. 5



$q, a, P=qa, M=qa^2$

$EI = const$

$\vartheta_A, \vartheta_B, y_B, y_D$ - ?