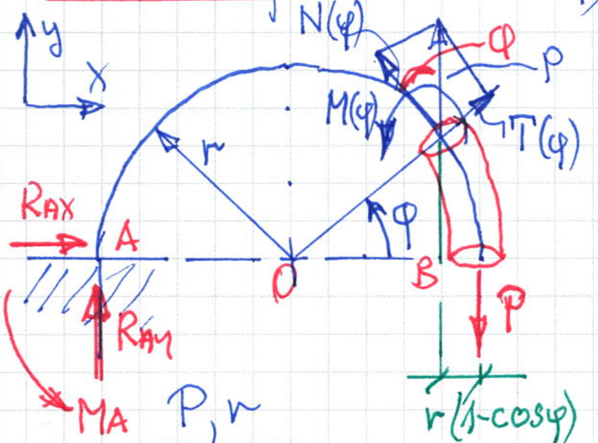
Curved bars; Castigliano's theorem

Exe 1

Internal forces $N(\varphi), T(\varphi), M(\varphi)$



P, r

$N(\varphi), T(\varphi), M(\varphi)$ - ?

The force system on the semicircular beam

I static eqs.

1. $\sum P_{ix} = R_{Ax} = 0$
2. $\sum P_{iy} = R_{Ay} - P = 0 \Rightarrow R_{Ay} = P$
3. $\sum M_i^A = -M_A + P \cdot 2r = 0 \Rightarrow$

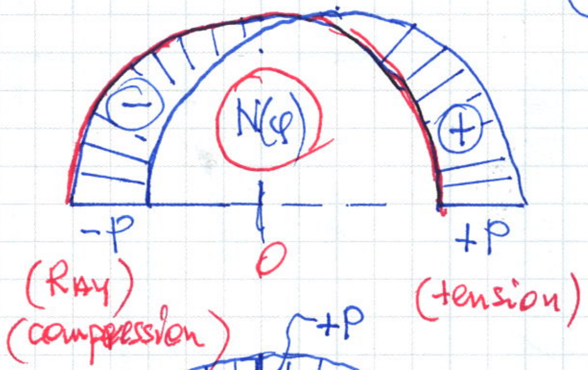
$$M_A = 2 \cdot P \cdot r$$

II Internal forces

a) $N(\varphi) = P \cdot \cos\varphi$

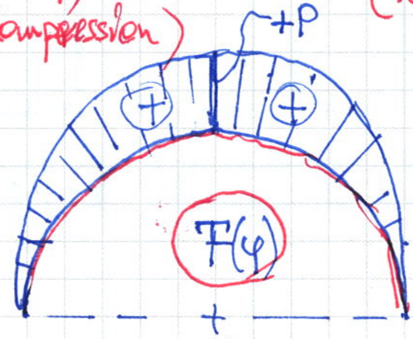
$N(0) = P$, $N\left(\frac{\pi}{2}\right) = 0$

$N(\pi) = -P$



b) $T(\varphi) = P \sin\varphi$

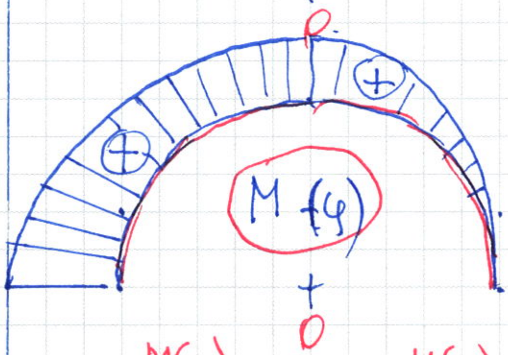
$T(0) = 0$, $T\left(\frac{\pi}{2}\right) = P$, $T(\pi) = 0$



c) $M(\varphi) = P(r - r \cdot \cos\varphi) = P \cdot r(1-\cos\varphi)$

$M(0) = 0$, $M\left(\frac{\pi}{2}\right) = P \cdot r$

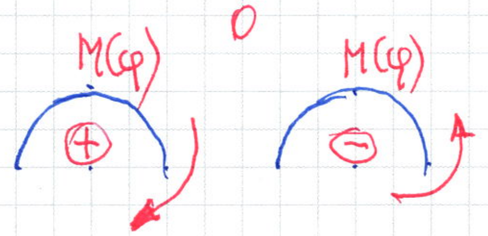
$M(\pi) = 2 \cdot P \cdot r$



$N(\pi) = -P = R_{Ay}$

$T(\pi) = 0 = R_{Ax}$

$M(\pi) = 2 \cdot P \cdot r = M_A$



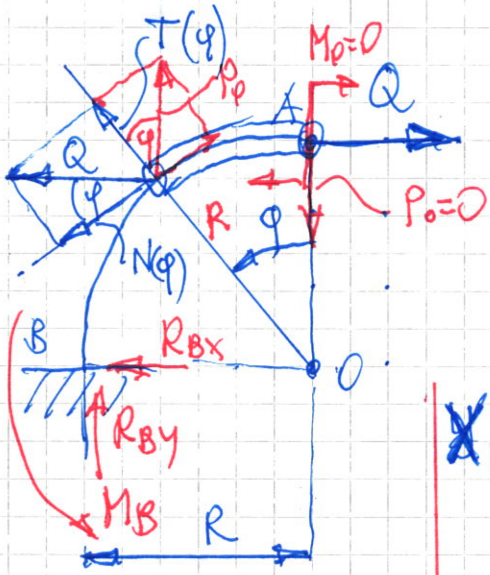
Ex. 2.

Q, R, $EI = const$, $EA = const$,

$x_A, y_A, \vartheta_A - ?$

Castigliano th.

$$x_A = \frac{\partial U}{\partial Q}, \quad y_A = \frac{\partial U}{\partial P_0}, \quad \vartheta_A = \frac{\partial U}{\partial M_0}$$



$$\cancel{x}_A = \frac{1}{EI}\int_S M(\varphi)\cdot\frac{\partial M(\varphi)}{\partial Q}ds + \frac{1}{EA}\int_S N(\varphi)\cdot\frac{\partial N(\varphi)}{\partial Q}ds$$

$$y_A = \frac{1}{EI}\int_S M(\varphi)\frac{\partial M(\varphi)}{\partial P_0}ds + \frac{1}{EA}\int_S N(\varphi)\frac{\partial N(\varphi)}{\partial P_0}ds$$

$ds = R \cdot d\varphi$

$$\vartheta_A = \frac{1}{EI}\int_S M(\varphi)\frac{\partial M(\varphi)}{\partial M_0}ds + \frac{1}{EA}\int_S N(\varphi)\frac{\partial N(\varphi)}{\partial M_0}ds$$

keep attention!
above, the arbitrary decision, that only bending and normal forces affect elastic energy

$$N(\varphi) = Q\cdot\cos\varphi - P_0\cdot\sin\varphi$$
$$T(\varphi) = Q\cdot\sin\varphi + P_0\cdot\cos\varphi$$
$$M(\varphi) = M_0 + P_0\cdot R\cdot\sin\varphi + Q\cdot R(1-\cos\varphi)$$

!

$$\frac{\partial M(\varphi)}{\partial Q} = R(1-\cos\varphi); \quad \frac{\partial M(\varphi)}{\partial P_0} = R\cdot\sin\varphi; \quad \frac{\partial M(\varphi)}{\partial M_0} = 1$$

$$\frac{\partial N(\varphi)}{\partial Q} = \cos\varphi; \quad \frac{\partial N(\varphi)}{\partial P_0} = -\sin\varphi; \quad \frac{\partial N(\varphi)}{\partial M_0} = 0$$

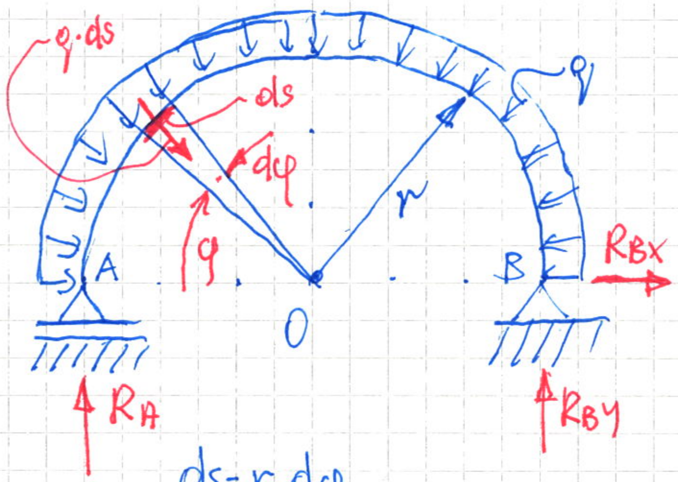
$$\frac{\partial T(\varphi)}{\partial Q} = \sin\varphi; \quad \frac{\partial T(\varphi)}{\partial P_0} = \cos\varphi, \quad \frac{\partial T(\varphi)}{\partial M_0} = 0$$

$$x_A = \frac{1}{EI}\int_0^{\frac{\pi}{2}}\left[\overset{=0}{M_0} + \overset{=0}{P_0}\cdot R\sin\varphi + Q\cdot R(1-\cos\varphi)\right]R(1-\cos\varphi)\cdot R\,d\varphi +$$

$$= \frac{1}{EA}\int_0^{\frac{\pi}{2}}\left[Q\cdot\cos\varphi - \overset{=0}{P_0}\sin\varphi\right]\cdot\cos\varphi\cdot R\,d\varphi = \frac{1}{EI}\int_0^{\frac{\pi}{2}} Q\cdot R^3(1-\cos\varphi)^2 d\varphi$$

$$+ \frac{1}{EA}\int_0^{\frac{\pi}{2}} Q\cos^2\varphi\cdot R\cdot d\varphi = ?$$

Ex.3.) Curved bars, continuous load



$ds = r\cdot d\varphi$

$\frac{q, r}{\text{reactions - ?}}$

(T) static eqs.

① $\sum P_{ix} = \int_0^{\pi} q\cdot\cos\varphi \underbrace{ds}_{r\cdot d\varphi} + R_{Bx} = 0$

$= q\cdot r\int_0^{\pi}\cos\varphi\, d\varphi + R_{Bx} = 0$

$q r \sin\varphi\Big|_0^{\pi} + R_{Bx} = 0$

$q r (0-0) + R_{Bx} = 0 \Rightarrow R_{Bx} = 0$

② $\sum P_{iy} = R_A + R_{By} - \int_0^{\pi} q r \sin\varphi\, d\varphi = 0$

$R_A + R_{By} + q r \cos\varphi\Big|_0^{\pi} = 0$

$R_A + R_{By} + q r(-1-1) = 0$

$R_A + R_{By} - 2qr = 0$

③ $\sum M_i^{O} = R_A\cdot r - R_{By}\cdot r = 0$

$R_A = R_{By}$

→ ② $2R_A - 2qr = 0 \Rightarrow$

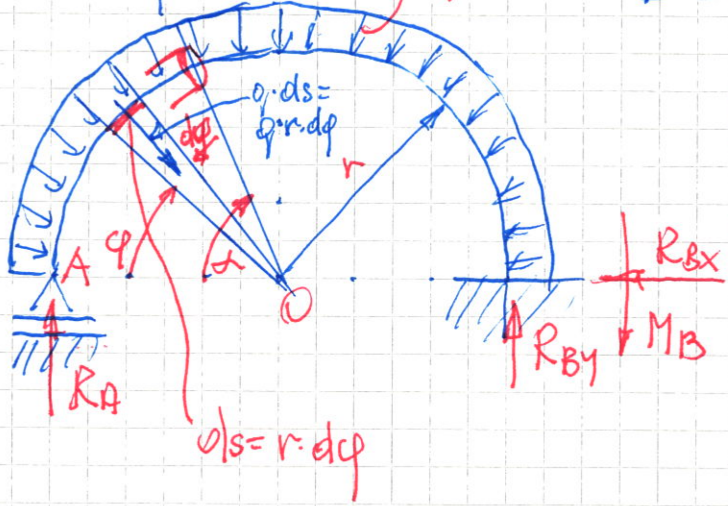
$\Rightarrow R_A = R_{By} = q\cdot r$

Ex. 4.

$q, r, EI = const, EA = const$

reactions - ?

Menabrea - Castigliano th.



I static eqs.

(1) $\sum P_{ix} = \int_0^\pi q\cdot\cos\varphi\cdot r\cdot d\varphi + R_{Bx} = 0$

(2) $\sum P_{iy} = R_A + R_{By} - \int_0^\pi q\cdot r\cdot\sin\varphi\, d\varphi = 0$

(3) $\sum M_i^O = R_A\cdot r - R_{By}\cdot r + M_B = 0$

4 reactions - 3 st. eqs. ⇒

⇒ 1x hyperstatic curved bar

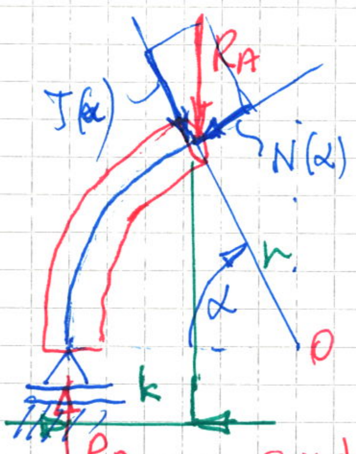
R_A - hyperstatic reaction

(4) geometrical eq.

$$\frac{\partial U}{\partial R_A} = y_A = 0$$

Menabrea - Castigliano th.

$$y_A = \frac{1}{EI}\int_0^\pi M(\alpha)\cdot\frac{\partial M(\alpha)}{\partial R_A}\, r\cdot d\varphi + \frac{1}{EA}\int_0^\pi N(\alpha)\frac{\partial N(\alpha)}{\partial R_A}\, r\, d\varphi = 0$$



II Internal forces

$$N(\alpha) = -R_A\cdot\cos\alpha - \int_0^\alpha q\cdot r\sin(\alpha-\varphi)\, d\varphi$$

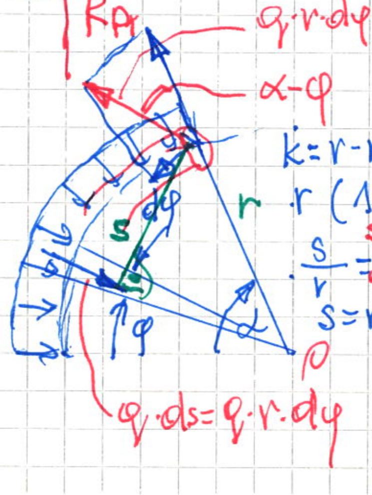
$$\frac{\partial N(\alpha)}{\partial R_A} = -\cos\alpha$$

$$M(\alpha) = -R_A\cdot k + \int_0^\alpha q\cdot r\cdot s\cdot d\varphi =$$

$$= -R_A\cdot r\cdot(1-\cos\alpha) + \int_0^\alpha q\cdot r\cdot r\cdot \overset{\sin}{\cancel{\cos}}(\alpha-\varphi)\, d\varphi =$$

$$= -R_A\cdot r(1-\cos\alpha) + q r^2\int_0^\alpha \overset{\sin}{\cancel{\cos}}(\alpha-\varphi)\, d\varphi$$

$$\frac{\partial M(\alpha)}{\partial R_A} = -r(1-\cos\alpha)$$

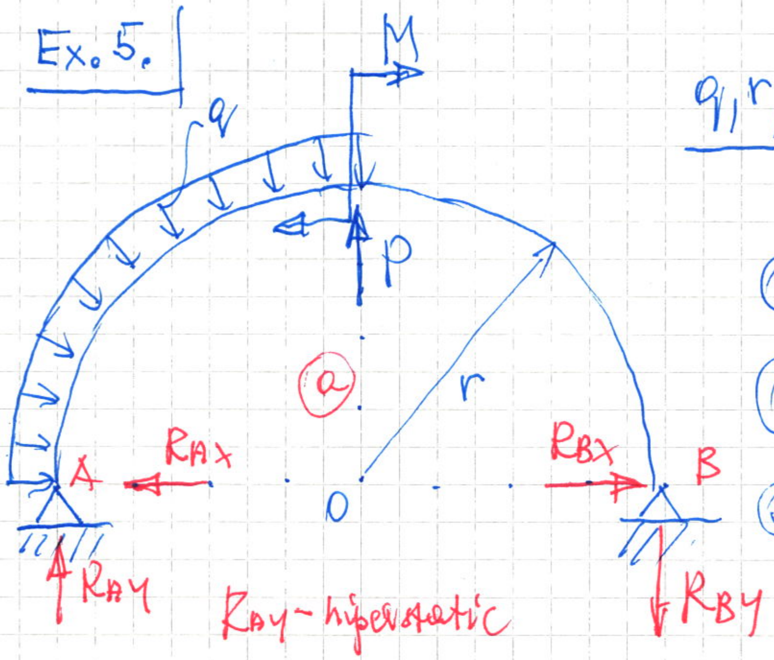


$k = r - r\cos\alpha = r(1-\cos\alpha)$

$\frac{s}{r} = \overset{\sin}{\cancel{\cos}}(\alpha-\varphi)$

$s = r\cdot\overset{\sin}{\cancel{\cos}}(\alpha-\varphi)$

Ex. 5.



$q, r, P = q \cdot r, M = q \cdot r^2$

reactions - ?

(1) Menabrea - Castigliano

(2) Maxwell - Mohr

(3) Force Method

for (a)

(I) static eqs

$$\left.\begin{array}{l} (1)\ \sum P_{ix} = 0 \\ (2)\ \sum P_{iy} = 0 \\ (3)\ \sum M_i^0 = 0 \end{array}\right\} \Rightarrow$$ 4 reactions - 3 st. eqs $\Rightarrow$ $\Rightarrow$ 1 × hyperstatic curved bar

Menabrea - Castigliano th.

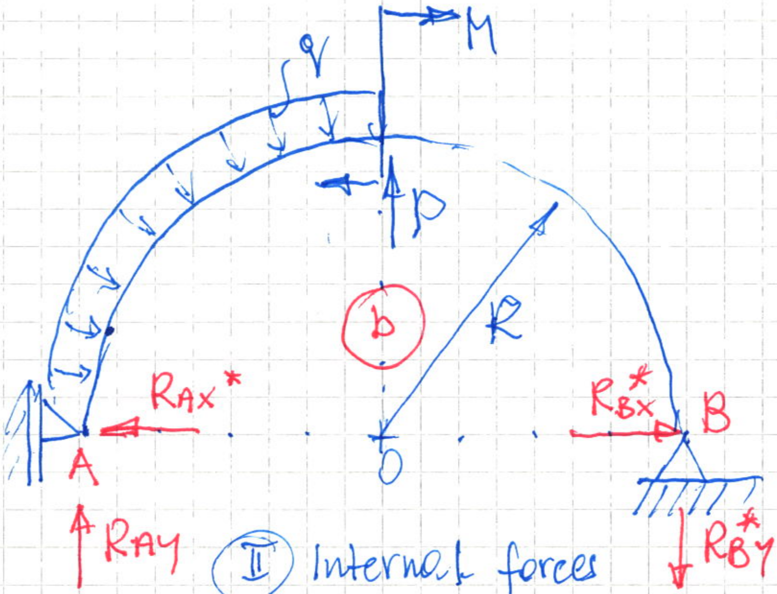
$$(4)\quad \frac{\partial U}{\partial R_{Ay}} = y_A = 0$$

$0 \le \varphi_1 \le \frac{\pi}{2}$	$\frac{\pi}{2} \le \varphi_2 \le \pi$
$M_1(\varphi) =$	$M_2(\varphi) = \ldots$
$N_1(\varphi) =$	$N_2(\varphi) =$

$$y_A = 0 = \frac{1}{EI}\left[\int_0^{\frac{\pi}{2}} M_1(\varphi)\cdot\frac{\partial M_1(\varphi)}{\partial R_{Ay}} r\,d\varphi + \int_{\frac{\pi}{2}}^{\pi} M_2(\varphi)\frac{\partial M_2(\varphi)}{\partial R_{Ay}} r\,d\varphi\right] +$$

$$+\frac{1}{EA}\left[\int_0^{\frac{\pi}{2}} N_1(\varphi)\cdot\frac{\partial N_1(\varphi)}{\partial R_{Ay}} r\,d\varphi + \int_{\frac{\pi}{2}}^{\pi} N_2(\varphi)\cdot\frac{\partial N_2(\varphi)}{\partial R_{Ay}} r\,d\varphi\right] = 0 \Rightarrow$$

$\Rightarrow R_{Ay} \Rightarrow$ (1), (2), (3) $\Rightarrow R_{Ax}, R_{Bx}, R_{By}$



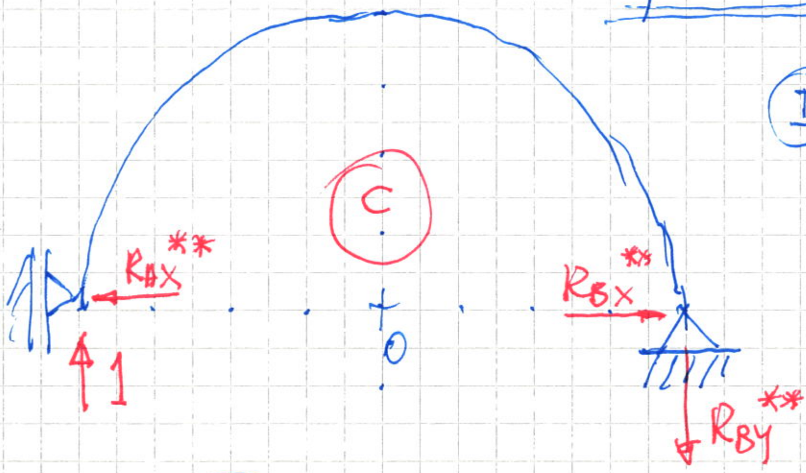
Maxwell-Mohr th.

I static eqs.

$$\left.\begin{aligned} &(1)\ \sum P_{ix} = 0 \\ &(2)\ \sum P_{iy} = 0 \\ &(3)\ \sum M_i^0 = 0 \end{aligned}\right\} \Rightarrow R_{Ax}^*, R_{Bx}^*, R_{By}^*$$

II Internal forces

$0 \le \varphi_1 \le \frac{\pi}{2}$	$\frac{\pi}{2}\ \varphi_2 \le \pi$
$M_1^*(\varphi) =$	$M_2^*(\varphi) =$
$N_1^*(\varphi) =$	$N_2^*(\varphi) =$



I static eqs.

$$\left.\begin{aligned} &(1)\ \sum P_{ix} = 0 \\ &(2)\ \sum P_{iy} = 0 \\ &(3)\ \sum M_i^0 = 0 \end{aligned}\right\} \Rightarrow R_{Ax}^{**}, R_{Bx}^{**}, R_{By}^{**}$$

II Internal forces

$ds = R \cdot d\varphi$

$0 \le \varphi_{1/2} \le \pi$

$M_{1/2}^0(\varphi) =$

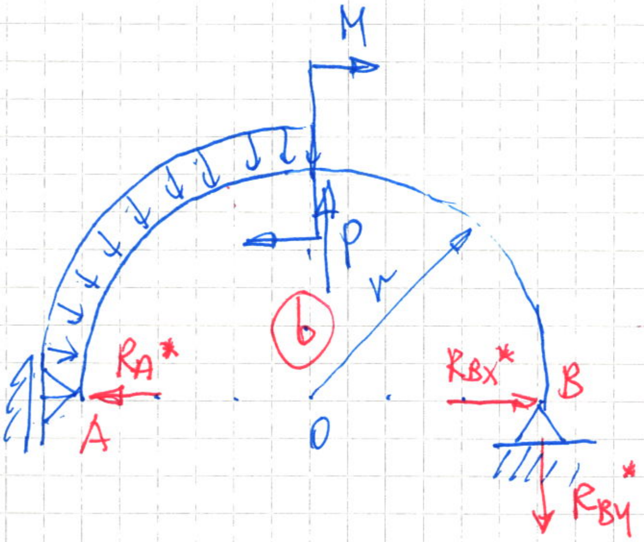
$N_{1/2}^0(\varphi) = 0$

$$(4)\quad y_A = \frac{1}{EI}\left[\int_0^{\frac{\pi}{2}} M_1^*(\varphi)\cdot M_{1/2}^0(\varphi)\, R\, d\varphi + \int_{\frac{\pi}{2}}^{\pi} M_2^*(\varphi)\, M_{1/2}^0(\varphi)\, R\, d\varphi\right] +$$

$$+ \frac{1}{EA}\left[\int_0^{\frac{\pi}{2}} N_1^*(\varphi)\cdot N_{1/2}^0(\varphi)\, R\, d\varphi + \int_{\frac{\pi}{2}}^{\pi} N_2^*(\varphi)\cdot N_{1/2}^0(\varphi)\, R\, d\varphi\right] = 0\ !$$

$$\rightarrow R_{Ay} \Rightarrow (1), (2), (3) \Rightarrow R_{Ax}, R_{Bx}, R_{By}$$

Force Method



Ⅰ static eqs.

① $\sum P_{ix} = 0$

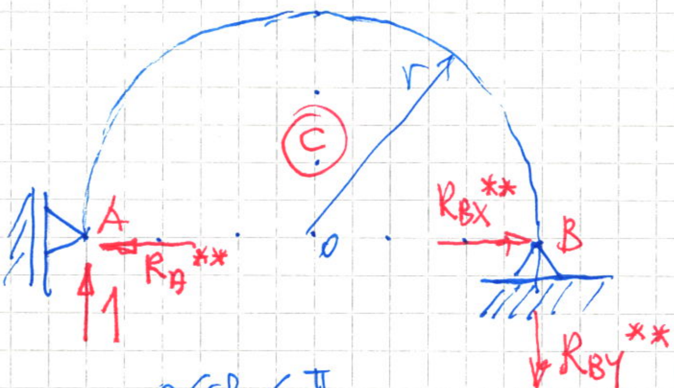
② $\sum P_{iy} = 0$

③ $\sum M_i^b = 0$

$\Rightarrow R_A^*, R_{Bx}^*, R_{By}^*$

Ⅱ Internal forces

$0 \le \varphi_1 \le \frac{\pi}{2}$	$\frac{\pi}{2} \le \varphi_2 \le \pi$
$M_1^*(\varphi) =$	$M_2^*(\varphi) =$
$N_1^*(\varphi) =$	$N_2^*(\varphi) =$



Ⅰ static eqs.

① $\sum P_{ix} = 0$

② $\sum P_{iy} = 0$

③ $\sum M_i = 0$

$R_A^{**}, R_{Bx}^{**}, R_{By}^{**}$

Ⅱ Internal forces

$0 \le \varphi_1 \le \frac{\pi}{2}$
$M_1^{**}(\varphi) =$
$N_1^{**}(\varphi) =$

$\frac{\pi}{2} \le \varphi_2 \le \pi$
$M_2^{**}(\varphi) =$
$N_2^{**}(\varphi) =$

$R_{Ay} = X_1 \Rightarrow$ hyperstatic

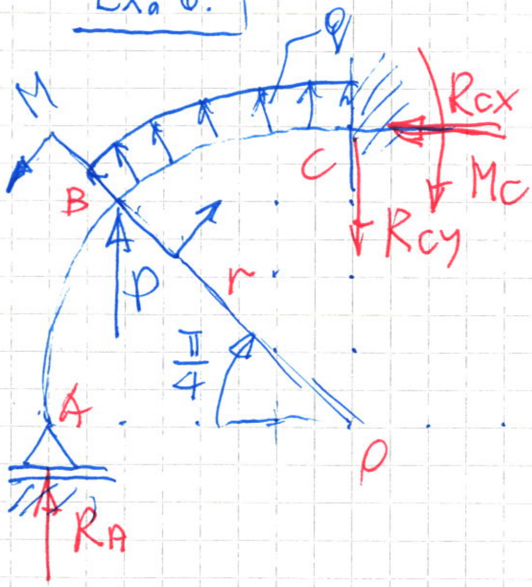
④ $X_1 \delta_{11} + \Delta_{1P} = 0$

$\Delta_{1P} \Rightarrow$ b + c

$\delta_{11} \Rightarrow$ c + c

$\Rightarrow X_1 = R_{Ay} \Rightarrow$ ①, ②, ③ for a $\Rightarrow R_{Ax}, R_{Bx}, R_{By}$

Ex. 6.



$q, r, P = q\cdot r, M = qr^2$

$EI = const, EA = const$

reactions - ?

① Menabrea - Castigliano th.

② Maxwell - Mohr method

③ Force method