

Maxwell-Mohr method

The Maxwell-Mohr procedure presents a universal method for computation of displacement at any point of any deformable structure.

For e.g. bending system, the Castigliano's theorem for computation of linear and angular displacement at point "i" may be presented as follows:

① Elastic energy

$$U = \sum_i \int_{(li)}^{} \frac{N_i^2}{2E_i A_i} dx_i + \sum_i \int_{(li)}^{} \frac{\beta_i T_i^2}{2G_i A_i} dx_i + \sum_i \int_{(li)}^{} \frac{M_i^2}{2E_i I_i} dx_i$$

② Displacements

$$p_i = \frac{\partial U}{\partial p_i} \Rightarrow y_i = \frac{\partial U}{\partial F_c}, N_i = \frac{\partial U}{\partial M}$$

Now suppose that at point "i" we apply a generalized force equal to 1.

This force will increase internal forces by respectively;

$$M^*, N^*, T^*$$

If the force is now a multiple of F of the force equal to 1, then this force will increase internal forces by respectively $F \cdot M^*$, $F \cdot N^*$, $F \cdot T^*$.

So if the force F is added to the mechanical system, the elastic energy will increase and take the form;

$$U = \sum_i \int_{(li)}^{} \frac{(N_i + F \cdot N_i^*)^2}{2E_i A_i} dx_i + \sum_i \int_{(li)}^{} \beta \frac{(T_i + F \cdot T_i^*)^2}{2G_i A_i} dx_i + \sum_i \int_{(li)}^{} \frac{(M_i + F \cdot M_i^*)^2}{2E_i I_i} dx_i$$

Let us now determine the displacement corresponding to the assumed force F

$\frac{y_i}{\delta_j} > f_i$ displacement

$$f = \frac{\partial U}{\partial F}$$

$$f = \sum_i \int \left(\frac{N_i + F \cdot N_i^*}{E_i A_i} \right) N_i^* dx_i + \sum_{i(i)} \int \phi_i \left(\frac{T_i + F \cdot T_i^*}{G_i A_i} \right) T_i^* dx_i +$$

$$\sum_{i(i)} \int \left(\frac{M_i + F \cdot M_i^*}{E_i I_i} \right) M_i^* dx$$

but the additional force $F=0$, because we don't want to change the loading conditions of our object (beam)

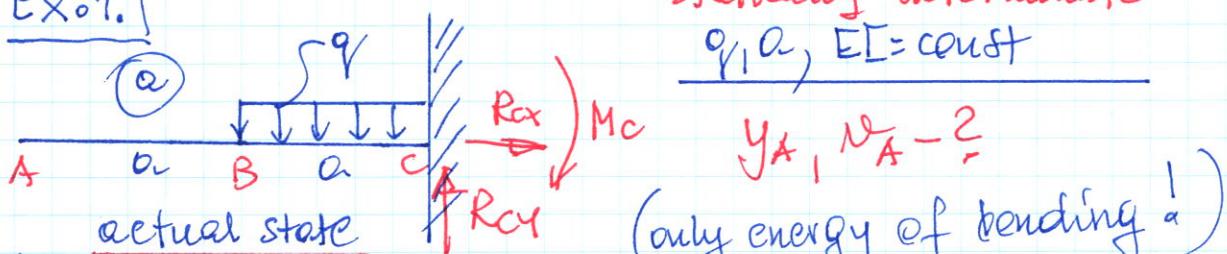
Then

$$f = \sum_{i(i)} \int \frac{N_i \cdot N_i^*}{E_i A_i} dx_i + \sum_{i(i)} \int \phi_i \frac{T_i \cdot T_i^*}{G_i A_i} dx_i + \sum_{i(i)} \int \frac{M_i \cdot M_i^*}{E_i I_i} dx_i$$

statically determinate

$$q_1 a, EI = \text{const}$$

Ex. 1.



$$0 \leq x_1 \leq a \quad M_1(x) = 0 \quad ; \quad a \leq x_2 \leq 2a \quad M_2(x) = -\frac{q}{2}(x-a)^2$$

$$1 \quad \textcircled{b} \quad \Rightarrow \quad 0 \leq x_1 \leq a \quad M_1^*(x) = -1 \cdot x = -x \quad \mid \quad a \leq x_2 \leq 2a \quad M_2^*(x) = -1 \cdot x = -x$$

unit state for y_A

$$1 \quad \textcircled{c} \quad \Rightarrow \quad 0 \leq x_1 \leq a \quad M_1^{**}(x) = -1 \quad \mid \quad a \leq x_2 \leq 2a \quad M_2^{**}(x) = -1$$

unit state for N_A

$$y_A \Rightarrow \textcircled{a} + \textcircled{b}$$

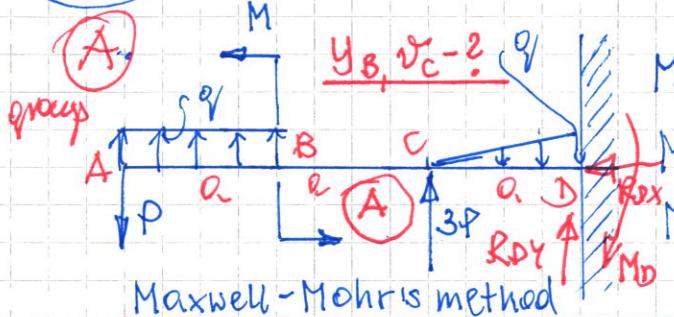
$$\begin{aligned}
 y_A &= \frac{1}{EI} \left[\int_0^a M_1(x) \cdot M_1^*(x) dx + \int_{2a}^{2a} M_2(x) \cdot M_2^*(x) dx \right] = ? \\
 &= \frac{1}{EI} \left[\int_0^a 0 \cdot (-x) dx + \int_a^{2a} -\frac{q(x-a)^2}{2}(-x) dx \right] = \\
 &= \frac{1}{EI} \int_0^a -\frac{q}{2} (x^2 - 2ax + a^2)(-x) dx = -\frac{q}{2EI} \int_0^a (-x^3 + 2x^2a - a^2x^2) dx = \\
 &= -\frac{q}{2EI} \left[-\frac{x^4}{4} + 2a \frac{x^3}{3} - a^2 \frac{x^2}{2} \right] \Big|_0^{2a} = \\
 &= -\frac{q}{2EI} \left(-\frac{16a^4}{4} + \frac{2a}{3} 8a^3 - \frac{a^2}{2} 4a^2 + \frac{a^4}{4} - \frac{2a}{3} a^3 + a^2 \frac{a^2}{2} \right) = \\
 &= -\frac{q}{2EI} \left(-4a^4 + \frac{16}{3} a^4 - 2a^4 + \frac{a^4}{4} - \frac{2}{3} a^4 + \frac{a^4}{2} \right) = \\
 &= \frac{7}{24} \frac{9a^4}{EI} = y_A
 \end{aligned}$$

$$\bar{N}_A \Rightarrow \textcircled{a} + \textcircled{c}$$

$$\begin{aligned}
 \bar{N}_A &= \frac{1}{EI} \left[\int_0^a M_1(x) \cdot M_1^{**}(x) dx + \int_{2a}^{2a} M_2(x) \cdot M_2^{**}(x) dx \right] = ? \\
 \bar{N}_A &= \frac{1}{EI} \left[\int_0^{2a} 0 \cdot (-1) dx + \int_a^{2a} -\frac{q(x-a)^2}{2} \cdot (-1) dx \right] = \\
 &= \frac{1}{EI} \int_0^{2a} -\frac{q}{2} (x^2 - 2ax + a^2)(-1) dx = \frac{q}{2EI} \int_0^{2a} (x^2 - 2ax + a^2) dx = \\
 &= \frac{q}{2EI} \left(\frac{x^3}{3} - 2a \frac{x^2}{2} + a^2 \cdot x \right) \Big|_0^{2a} = \frac{q}{2EI} \left(\frac{8a^3}{3} - 4a^3 + 2a^3 \right) = \\
 &\quad - \frac{q^3}{3} + a^3 - a^3 = \frac{q}{2EI} \left(\frac{8a^3}{3} - 4a^3 + 2a^3 - \frac{q^3}{3} \right) = \\
 &= \frac{q}{2EI} \left(\frac{8}{3} - \frac{1}{3} - 2 \right) = \frac{9a^3}{6EI} = \bar{N}_A
 \end{aligned}$$

Ex 1.

$E, I = \text{const}$



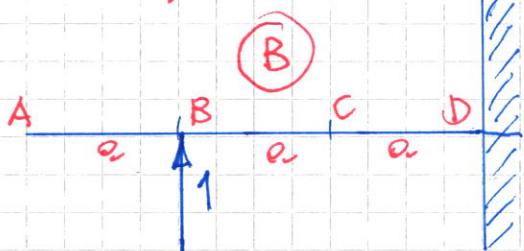
$$M_1(x) = -P \cdot x + \frac{q x^2}{2}$$

$$M_2(x) = -P \cdot x + q a \left(x - \frac{a}{2} \right) - M$$

$$M_3(x) = -P \cdot x + q a \left(x - \frac{a}{2} \right) - M + 3P(x-2a) - \frac{q(x-2a)^3}{6a}$$

Maxwell-Mohr's method

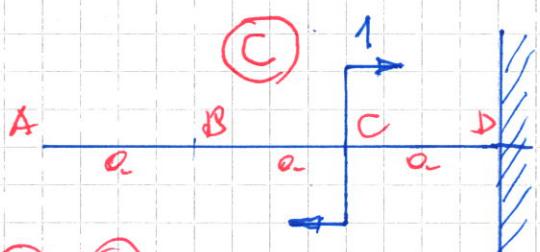
statically determinate



$$M_1^{(*)}(x) = 0$$

$$M_2^{(*)}(x) = 1 \cdot (x-a) = x-a$$

$$M_3^{(*)}(x) = 1 \cdot (x-a) = x-a$$



$$M_1^{(**)}(x) = 0$$

$$M_2^{(**)}(x) = 0$$

$$M_3^{(**)}(x) = 1$$

(A) + (B)

$$y_B = \frac{1}{EI} \left[\int_{0}^{a} M_1(x) \cdot M_1^{(*)}(x) dx + \int_{a}^{2a} M_2(x) \cdot M_2^{(*)}(x) dx + \int_{2a}^{3a} M_3(x) \cdot M_3^{(*)}(x) dx \right] =$$

$$= \frac{1}{EI} \left[\int_{0}^{a} \left(-P \cdot x + \frac{q x^2}{2} \right) \cdot 0 \cdot dx + \int_{a}^{2a} \left[-P \cdot x + q a \left(x - \frac{a}{2} \right) - M \right] (x-a) dx + \int_{2a}^{3a} \left[-P \cdot x + q a \left(x - \frac{a}{2} \right) - M + 3P(x-2a) \right] (x-a) dx \right]$$

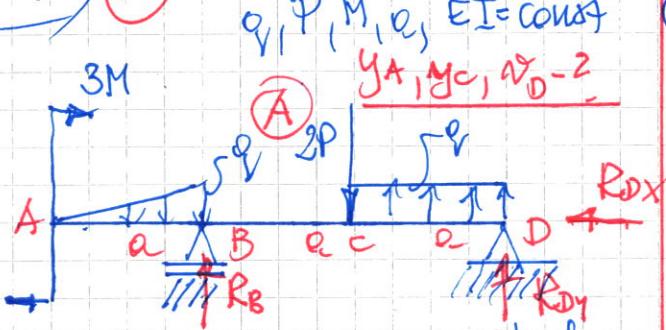
(A) + (C)

$$w_C = \frac{1}{EI} \left[\int_{0}^{a} M_1(x) \cdot M_1^{(**)}(x) dx + \int_{a}^{2a} M_2(x) \cdot M_2^{(**)}(x) dx + \int_{2a}^{3a} M_3(x) \cdot M_3^{(**)}(x) dx \right] =$$

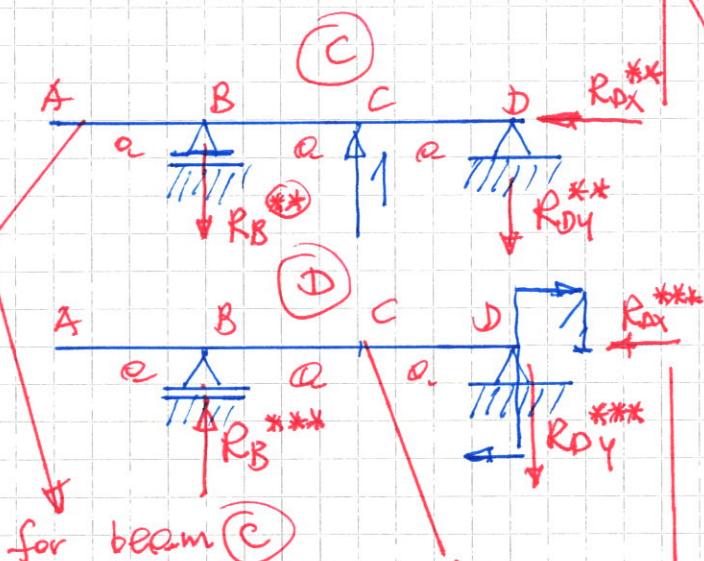
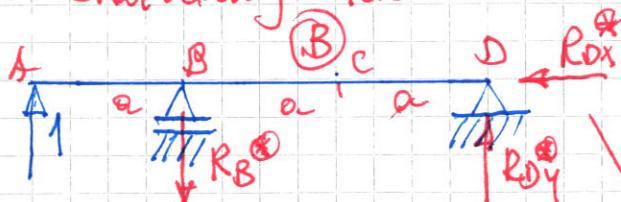
$$\frac{1}{EI} \left[\int_{0}^{a} \left(-P \cdot x + \frac{q x^2}{2} \right) \cdot 0 dx + \int_{a}^{2a} \left[-P \cdot x + q a \left(x - \frac{a}{2} \right) - M \right] \cdot 0 dx + \int_{2a}^{3a} \left[-P \cdot x + q a \left(x - \frac{a}{2} \right) - M + 3P(x-2a) - \frac{q(x-2a)^3}{6a} \right] \cdot 1 dx \right]$$

Ex. 2.

(A)



Maxwell-Mohr's method
statically determinate



for beam (C)

$$\begin{aligned} ① \sum P_{ix} &= -R_{DX}^{**} = 0 \\ ② \sum P_{iy} &= -R_B^{**} + 1 - R_{Dy}^{**} = 0 \\ ③ \sum M_i^{\phi} &= -R_B^{**} \cdot 2a + 1 \cdot x = 0 \end{aligned}$$

$\Downarrow R_B^{**}, R_{DX}^{**}, R_{Dy}^{**}$

$$\begin{aligned} M_1^o(x) &= 0 \\ M_2^o(x) &= -R_B^{**}(x-a) \\ M_3^o(x) &= -R_B^{**}(x-a) + 1(x-2a) \end{aligned}$$

for beam (A)

$$① \sum P_{ix} = -R_{DX} = 0$$

$$② \sum P_{iy} = -\frac{1}{2}qa + R_B - 2P + qa + R_{Dy} = 0$$

$$\begin{aligned} ③ \sum M_i^{\phi} &= 3M - \frac{1}{2}qa \cdot \frac{2}{3}a + R_B \cdot 2a \\ -2P \cdot a + qa \cdot \frac{2}{3}a &= 0 \end{aligned}$$

$\Downarrow R_B, R_{DX}, R_{Dy}$

$$0 \leq x_1 \leq a$$

$$M_1(x) = 3M - \frac{qX^3}{6a}$$

$$a \leq x_2 \leq 2a$$

$$M_2(x) = 3M - \frac{1}{2}qa(x - \frac{2}{3}a) + R_B(x-a)$$

$$2a \leq x_3 \leq 3a$$

$$\begin{aligned} M_3(x) &= 3M - \frac{1}{2}qa(x - \frac{2}{3}a) + R_B(x-a) \\ -2P(x-2a) + q(x-2a)^2 &= 0 \end{aligned}$$

for beam (B)

$$① \sum P_{ix} = 1 - R_B^{**} + R_{Dy}^{**} = 0$$

$$② \sum P_{iy} = -R_D^{**} = 0$$

$$③ \sum M_i^{\phi} = 1 \cdot 3a - R_B^{**} \cdot 2a = P$$

$\Downarrow R_B^{**}, R_{DX}^{**}, R_{Dy}^{**}$

$$M_1^o(x) = 1 \cdot x = x$$

$$M_2^o(x) = 1 \cdot x - R_B^{**}(x-a)$$

$$x - R_B^{**} \cdot x + R_B^{**} \cdot a$$

$$M_3^o(x) = x - R_B^{**} \cdot x + R_B^{**} \cdot a$$

for beam D

$$\sum P_{ix} = -R_{DX}^{***} = 0$$

$$\sum P_{iy} = R_B^{***} - R_{Dy}^{***} = 0$$

$$\sum M_i^{\phi} = R_B^{***} \cdot 2a + 1 = 0$$

$\Downarrow R_B^{***}, R_{DX}^{***}, R_{Dy}^{***}$

$$M_1^o(x) = 0$$

$$M_2^o(x) = R_B^{***}(x-a)$$

$$M_3^o(x) = R_B^{***}(x-a)$$

$$M_3^o(x) = M_2^o(x)$$

$$y_A = \frac{1}{EI} \left[\int_0^a M_1(x) \cdot M_1^o(x) dx + \int_a^{2a} M_2(x) \cdot M_2^o(x) dx + \int_{2a}^{3a} M_3(x) \cdot M_3^o(x) dx \right] = ?$$

y_A $(A + B)$

$\overset{a}{\underset{0}{\int}}$ $\overset{2a}{\underset{a}{\int}}$ $\overset{3a}{\underset{2a}{\int}}$

$$y_C = \frac{1}{EI} \left[\int_0^a M_1(x) \cdot M_1^o(x) dx + \int_a^{2a} M_2(x) \cdot M_2^o(x) dx + \int_{2a}^{3a} M_3(x) \cdot M_3^o(x) dx \right] = ?$$

y_C $(A + C)$

$\overset{a}{\underset{0}{\int}}$ $\overset{2a}{\underset{a}{\int}}$ $\overset{3a}{\underset{2a}{\int}}$

$$\delta_D = \frac{1}{EI} \left[\int_0^a M_1(x) \cdot M_1^o(x) dx + \int_a^{2a} M_2(x) \cdot M_2^o(x) dx + \int_{2a}^{3a} M_3(x) \cdot M_3^o(x) dx \right] = ?$$

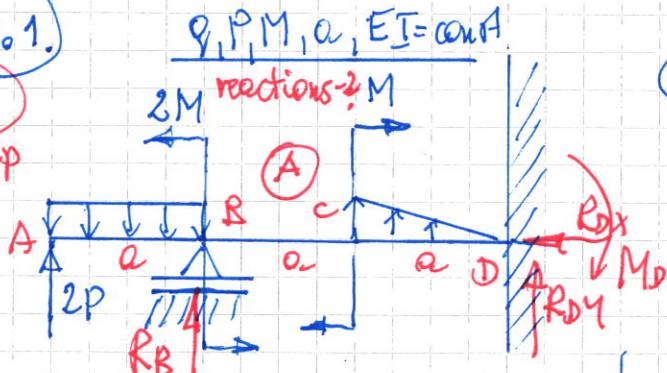
δ_D $(A + D)$

$\overset{a}{\underset{0}{\int}}$ $\overset{2a}{\underset{a}{\int}}$ $\overset{3a}{\underset{2a}{\int}}$

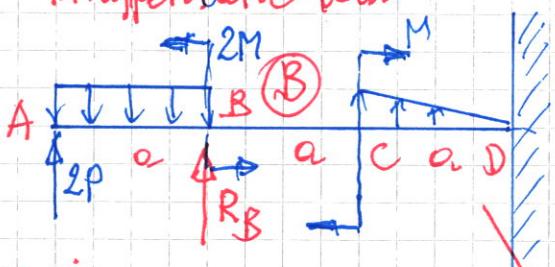
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Ex. 1.

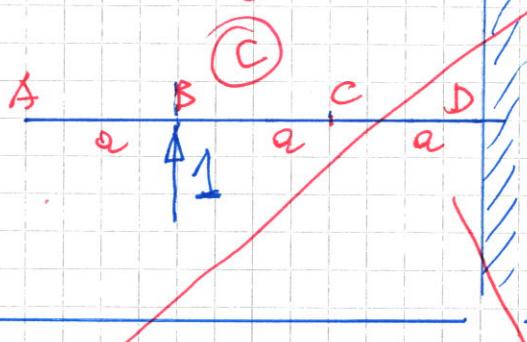
(B)
group



Maxwell-Mohr's method
(analytical version)
1x hyperstatic beam



Note: now statically determinate, but with deflection $y_B = 0$!



Geometrical equation again

$$y_B = \frac{1}{EI} \left[\int_0^{2a} M_1(x) \cdot M_1^*(x) dx + \int_a^{3a} M_2(x) \cdot M_2^*(x) dx + \int_{2a}^{2a} M_3(x) \cdot M_3^*(x) dx \right]$$

I static eqs.

$$\textcircled{1} \sum P_i x = -R_D x = 0$$

$$\textcircled{2} \sum P_i y = 2P - qa + R_B + \frac{qa}{2} + R_D y = 0$$

$$\textcircled{3} \sum M_i P = 2P \cdot 3a - qa \left(\frac{5}{2}a \right) - 2M + R_B \cdot 2a + M + \frac{1}{2}qa \left(\frac{2}{3}a \right) + M_D = 0$$

$R_B, R_D X, R_D Y, M_D$ - 4 reactions

4 reactions - 3 static eqs. \Rightarrow 1x hyperst.

R_B - hyperstatic reaction

II Geometrical eq.

$$\textcircled{4} y_B = 0$$

Bending moments for beam (B)

$$M_1(x) = 2P \cdot x - \frac{qx^2}{2}$$

$$M_2(x) = 2P \cdot x - qa \left(x - \frac{a}{2} \right) - 2M + R_B(x-a)$$

$$M_3(x) = 2P \cdot x - qa \left(x - \frac{a}{2} \right) - 2M$$

$$+ R_B(x-a) + M + \frac{q}{2} \frac{(x-a)^2}{2} - \frac{q}{6} \frac{(x-a)^3}{3}$$

trapeze

Bending moments for beam (C)

$$M_1^*(x) = 0$$

$$M_2^*(x) = 1(x-a) = (x-a)$$

$$M_3^*(x) = 1(x-a) = x-a$$

$$\Rightarrow y_B = 0$$

$$y_B = \frac{1}{EI} \left[\int_0^a (2P \cdot x - \frac{qx^2}{2}) \cdot 0 dx + \int_a^{2a} [2P \cdot x - qa(x - \frac{a}{2}) - 2M + R_B(x-a) + M + \frac{q}{2} \frac{(x-a)^2}{2} - \frac{q}{6} \frac{(x-a)^3}{3}] \cdot (x-a) dx \right]$$

$$\Rightarrow R_B = 0 \quad \text{from } \textcircled{1}, \textcircled{2}, \textcircled{3} \text{ static eqs.} \Rightarrow R_D X, R_D Y, M_D$$