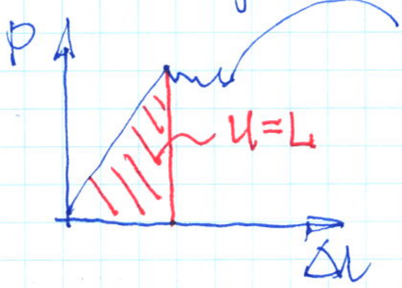
Canonical Equations of Force Method
(Równanie kanoniczne metody sił)

Canonical equations of force method offer a unified procedure for analysis of statically indeterminate structures of different types. The word "canonical" indicates that this equations are presented in standard or in an orderly fashion form. Very important is that canonical equations of the force method may be presented in a matrix form.



P_i – generalized force

p_i – generalized displacement

for tension-compression

$$L = U = L = \frac{1}{2} P \cdot \Delta l \; ; \quad \Delta l = \frac{P \cdot l}{EA}$$

$$L = U = L = \frac{1}{2} P \cdot \frac{Pl}{EA} = \frac{1}{2} \frac{P^2 l}{EA}$$

(energy is the square function of force)

More generally

$$L_i = U_i = \frac{1}{2} P_i \cdot p_i \qquad L = U = \frac{1}{2} \sum_i P_i p_i$$

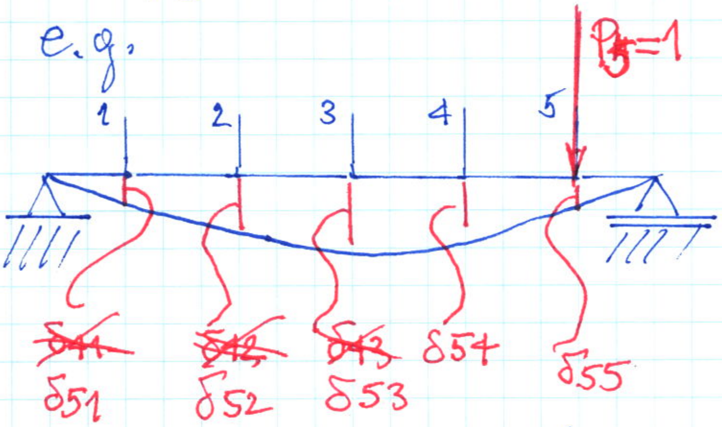
but $\boxed{p_i = \sum_{k=1}^{n} P_k \cdot \delta_{ik}}$ — in direction "i" (first index)

Coefficient δ_{ik} is called the <u>unit displacement</u> since it is caused by unit primary generalized force $P_k = 1$ (k – second index)

$$\boxed{\underset{\text{unit displacement}}{\delta_{ik}} = \frac{\text{generalized displacement}}{\text{generalized force}}}$$

(liczba wpływowa)

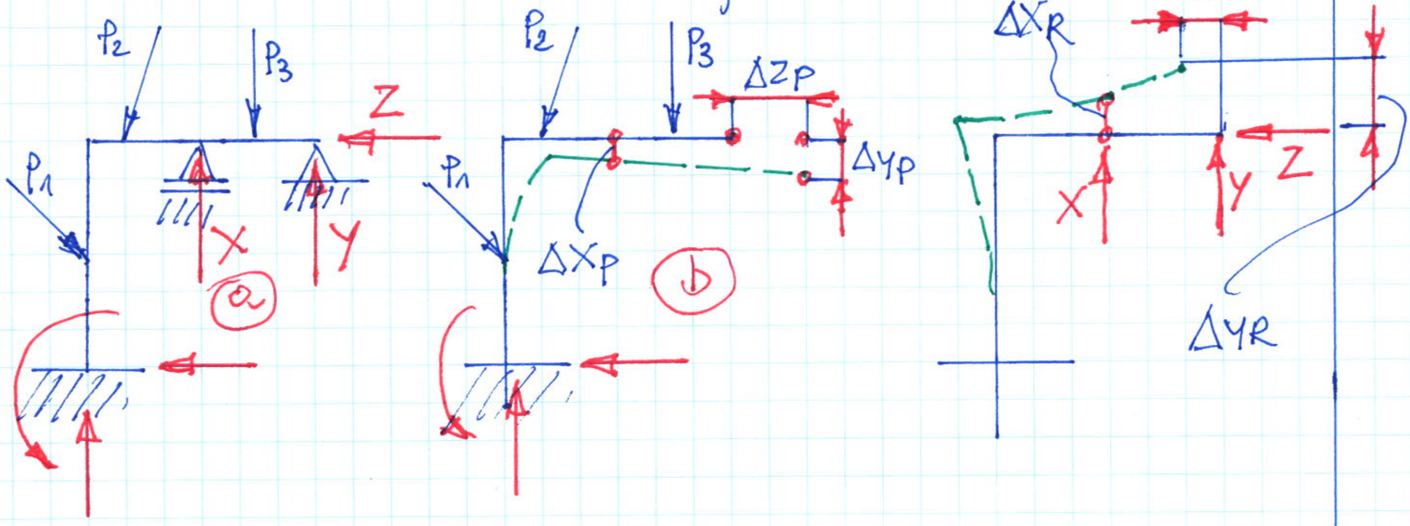
e.g.



$\delta_{51} = \delta_{15}$

$\delta_{ik} = \delta_{ki}$

Let us consider a simple redundant structure (hyperstatic) such as frame. The number of redundant is 3



$\Delta X_P, \Delta Y_P, \Delta Z_P$ – displacements caused by active forces, respectively: P_1, P_2, P_3 for points where the are hyperstatic forces, respectively: X, Y, Z

$\Delta X_R, \Delta Y_R, \Delta Z_R$ – displacements caused by ~~active~~ passive forces (hyperstatic reactions) respectively: X, Y, Z for points where are these hyperstatic reactions

$$\boxed{U = L = \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n} P_i \cdot P_k \cdot \delta_{ik}}$$

$$U = \frac{1}{2}\Big(X^2\delta_{xx} + Y^2\delta_{yy} + Z^2\delta_{zz} + P_1^2\delta_{11} + P_2^2\delta_{22} + P_3^2\delta_{33}\Big) +$$
$$+ X\cdot Y\cdot\delta_{xy} + Y\cdot Z\delta_{yz} + Z\cdot X\cdot\delta_{zx} + XP_1\delta_{x1} + XP_2\delta_{x2} + X\cdot P_3\delta_{x3}$$
$$+ Y\cdot P_1\delta_{y1} + Y\cdot P_2\cdot\delta_{y2} + Y\cdot P_3\delta_{y3} + Z\cdot P_1\delta_{z1} + Z\cdot P_2\delta_{z2} + Z\cdot P_3\delta_{z3}$$
$$+ P_1\cdot P_2\delta_{12} + P_2\cdot P_3\cdot\delta_{23} + P_3\cdot P_1\cdot\delta_{31}$$

but principle of minimum energii

$$\frac{\partial U}{\partial X} = 0, \quad \frac{\partial U}{\partial Y} = 0, \quad \frac{\partial U}{\partial Z} = 0$$

$$\frac{\partial U}{\partial X} = X\delta_{xx} + Y\delta_{xy} + Z\delta_{xz} + P_1\delta_{x1} + P_2\delta_{x2} + P_3\delta_{x3} = 0$$

$$\frac{\partial U}{\partial Y} = X\delta_{yx} + Y\delta_{yy} + Z\delta_{yz} + P_1\delta_{y1} + P_2\delta_{y2} + P_3\delta_{y3} = 0$$

$$\frac{\partial U}{\partial Z} = X\delta_{zx} + Y\delta_{zy} + Z\delta_{zz} + P_1\delta_{z1} + P_2\delta_{z2} + P_3\delta_{z3} = 0$$

Oznaczmy:

$$P_1\delta_{x1} + P_2\delta_{x2} + P_3\delta_{x3} = \Delta_{XP}$$
$$P_1\delta_{y1} + P_2\delta_{y2} + P_3\delta_{y3} = \Delta_{YP}$$
$$P_1\delta_{z1} + P_2\delta_{z2} + P_3\delta_{z3} = \Delta_{ZP}$$

przemieszczenia wywołane działaniem sił czynnych P_1, P_2, P_3

displacements caused by active forces P_1, P_2, P_3

Z kolei:

displacements caused by hyperstatic forces X, Y, Z

$$X\delta_{xx} + Y\delta_{xy} + Z\delta_{xz} = \Delta_{XR}$$
$$X\delta_{yx} + Y\delta_{yy} + Z\delta_{yz} = \Delta_{YR}$$
$$X\delta_{zx} + Y\delta_{zy} + Z\delta_{zz} = \Delta_{ZR}$$

Ostatecznie

Canonical equations of Force Method for X, Y, Z

$$\begin{cases} \overbrace{X\delta_{xx} + Y\delta_{xy} + Z\delta_{xz}}^{\Delta X_R} + \Delta_{XP} = 0 \\ X\delta_{yx} + Y\delta_{yy} + Z\delta_{yz} + \Delta_{YP} = 0 \\ X\delta_{zx} + Y\delta_{zy} + Z\delta_{zz} + \Delta_{ZP} = 0 \end{cases}$$

równanie kanoniczne metody sił dla ramy

Można dalej: $X = X_1$, $Y = X_2$, $Z = X_3$, itd.

If we have more hyperstatic forces: $X_1, \dots, X_n$

bardziej ogólnie

More generally

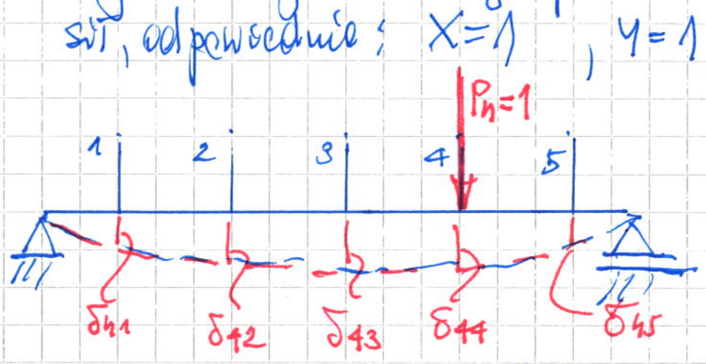
Canonical equations for $X_1 \dots X_n$ forces

$$\begin{cases} \delta_{11}X_1 + \delta_{12}X_2 + \delta_{13}\cdot X_3 + \dots + \delta_{1n}X_n + \Delta_{1P} = 0 \\ \delta_{21}X_1 + \delta_{22}X_2 + \delta_{23}\cdot X_3 + \dots + \delta_{2n}X_n + \Delta_{2P} = 0 \\ \vdots \\ \delta_{n1}X_1 + \delta_{n2}X_2 + \delta_{n3}X_3 + \dots + \delta_{nn}X_n + \Delta_{nP} = 0 \end{cases}$$

układ równań kanonicznych metody sił

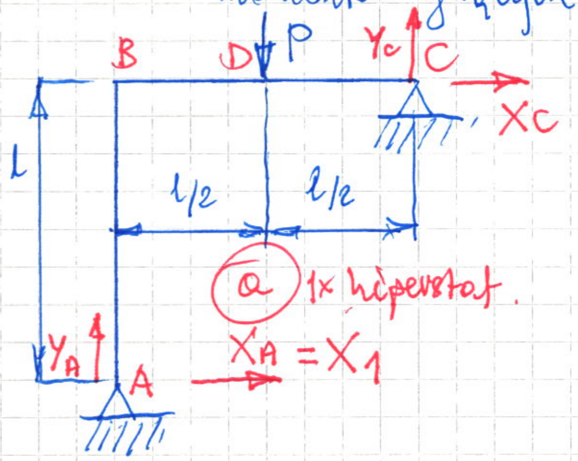
Unit displacements

Liczby wpływowe – jako odpowiednie przemieszczenie układu statycznie wyznaczalnego pod obciążeniem działaniem sił, odpowiednio: $X = 1$, $Y = 1$, $Z = 1$



$\delta_{41} = \delta_{14}$, itd.

Ex. 1. Stosując metodę sił (force method) wyznaczyć wykres momentów gnących dla ramy (a)



- rama jednokrotnie hiperst.

I warunki statyki (static eqs.)

(1) $\sum P_{ix} = X_A + X_C = 0 \qquad X_A = -X_C$

(2) $\sum P_{iy} = Y_A - P + Y_C = 0$

(3) $\sum M_i^C = Y_A \cdot l - X_A \cdot l - P \cdot \frac{l}{2} = 0$

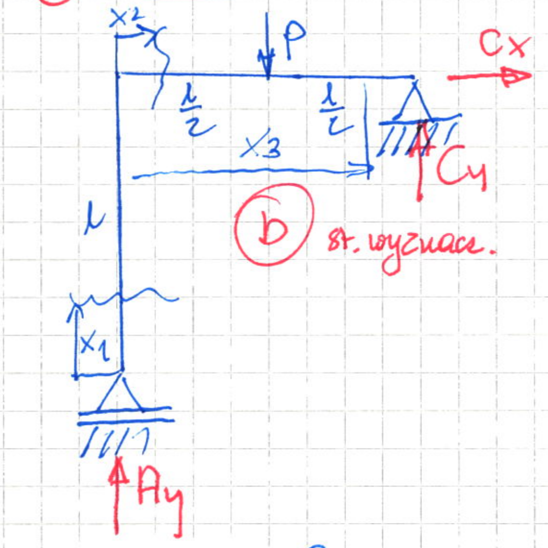
4 reakcje - 3 warunki statyki

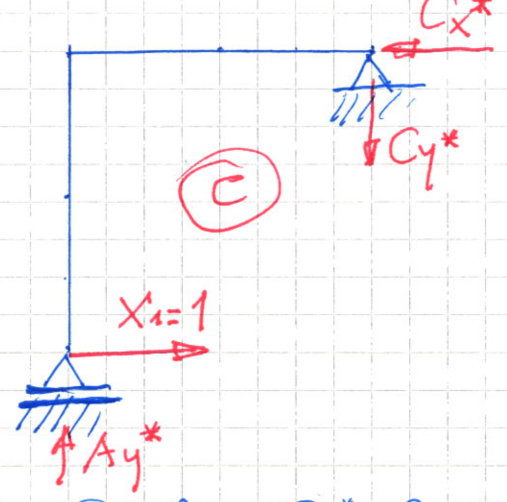
$\Rightarrow$ 1x hiperstat.

$X_A = X_1$ - wielkość hiperstatyczna (hyperstatic force)

Równanie kanoniczne metody sił w tym przypadku

(4) $\boxed{X_1 \delta_{11} + \Delta_{1p} = 0}$ (tylko energia od zginania, bez rozciągania/ściskania)





(1) $\sum P_{ix} = C_x = 0$

(2) $\sum P_{iy} = A_y - P + C_y = 0$

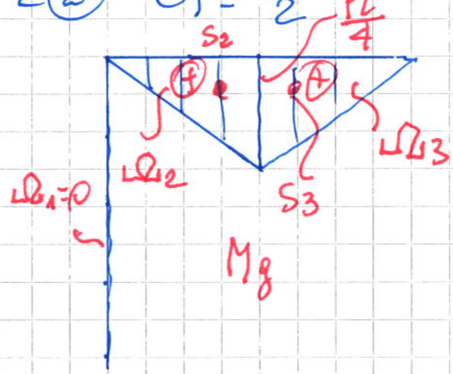
(3) $\sum M_i^C = A_y \cdot l - P \cdot \frac{l}{2} = 0 \Rightarrow A_y = \frac{P}{2}$

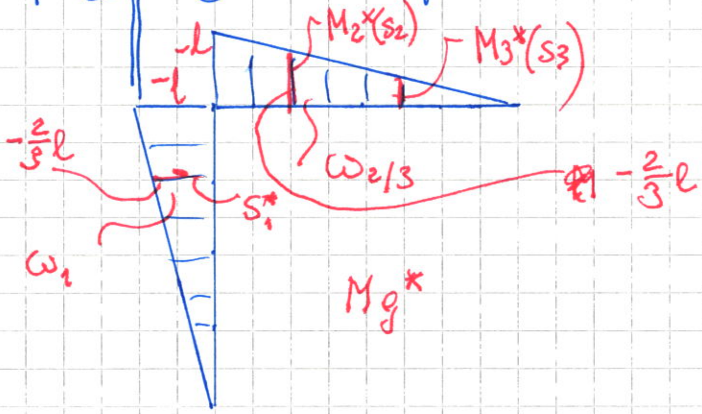
z (2) $C_y = \frac{P}{2}$

(1) $\sum P_{ix} = \cancel{C_x^* = 0} \quad 1 - C_x^* = 0 \Rightarrow C_x^* = 1$

(2) $A_y^* - C_y^* = 0$

(3) $\sum M_i^C = A_y^* \cdot l - 1 \cdot l = 0 \quad A_y^* = 1 = C_y^*$





$M_1(x) = 0$

$M_2(x) = A_y \cdot x = \frac{P}{2} \cdot x$

$M_2(0) = 0, \quad M_2\left(\frac{l}{2}\right) = \frac{P}{2} \cdot \frac{l}{2} = \frac{Pl}{4}$

$M_1^*(x) = -1 \cdot x = -x$

Do wyznaczenia : Δ_{1P} , δ_{11} zastosować metodę MM, sposób Wereszczagina

$\Delta_{1P} \Rightarrow b + c$

$$\Delta_{1P} = \frac{1}{EI}\left[\int_0^l M_1(x)\cdot M_1^*(x)dx + \int_0^{\frac{l}{2}} M_2(x)\cdot M_2^*(x)\,dx + \int_{\frac{l}{2}}^{l} M_3(x)\cdot M_3^*(x)dx\right]$$

$$= \frac{1}{EI}\left[\Omega_1\cdot M_1^*(s_1) + \Omega_2\cdot M_2^*(s_2) + \Omega_3\cdot M_3^*(s_3)\right] =$$

$$\Omega_1 = 0 \quad , \quad \Omega_2 = \Omega_3 = \frac{1}{2}\cdot\frac{Pl}{4}\cdot\frac{l}{2} = \frac{Pl^2}{16}$$

$$M_1^*(s_1) = l \quad , \quad M_2^*(s_2) = -\frac{2}{3}l \quad , \quad M_3^*(s_3) = -\frac{1}{3}l$$

$$\Delta_{1P} = \frac{1}{EI}\left[0 + \frac{Pl^2}{16}\cdot\left(-\frac{2}{3}l\right) + \frac{Pl^2}{16}\left(-\frac{1}{3}\right)\right] = -\frac{Pl^3}{16EI}$$

$\delta_{11} \Rightarrow c + c$

$~~\delta_{11}~~ =$ $\quad \omega_1 = \omega_{2/3} = \frac{1}{2}\cdot l\,(-l) = -\frac{l^2}{2}$

$$\delta_{11} = \frac{1}{EI}\left[\omega_1\cdot Mg_1^*(s_1) + \omega_{2/3}\cdot Mg_{2/3}^*(s_{2/3})\right] =$$

$$= \frac{1}{EI}\left[\left(-\frac{l^2}{2}\right)\cdot\left(-\frac{2}{3}l\right) + \left(-\frac{l^2}{2}\right)\cdot\left(-\frac{2}{3}l\right)\right] = \frac{2}{3}\frac{l^3}{EI}$$

(4) $\quad X_1\cdot\frac{2}{3}\frac{l^3}{EI} - \frac{Pl^3}{16EI} = 0 \quad \Big|\cdot 48EI$

$32X_1 - 3P = 0 \quad \Rightarrow \quad X_1 = X_A = \frac{3}{32}P$

Z warunków statyki

z (1) $\quad X_A + X_C = 0 \quad \Rightarrow \quad X_C = -X_A = -\frac{3}{32}P$

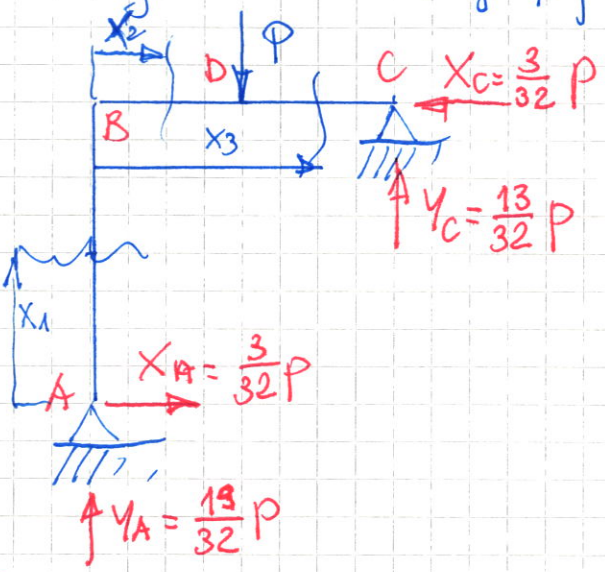
z (3) $\quad Y_A\cdot l - X_A\cdot l - P\frac{l}{2} = 0$

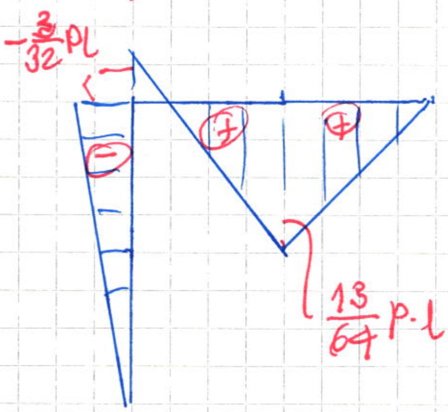
$Y_A\cdot l - \frac{3}{32}P\cdot l - \frac{Pl}{2} = 0 \qquad Y_A\cdot l = \frac{Pl}{2} + \frac{3Pl}{32} = \frac{19}{32}P\cdot l$

$Y_A = \frac{19}{32}P$

z (2) $\quad Y_A - P + Y_C = 0 \qquad Y_C = P - Y_A = P - \frac{19}{32}P = \frac{13}{32}P$

Wykres momentów gnących





$0 \le x_1 \le l$

$M_1(x) = -X_A \cdot x$

$M_1(0) = 0, \quad M_1(l) = -\frac{3}{32}P\cdot l$

$0 \le x_2 \le \frac{l}{2}$

$M_2(x) = -\frac{3}{32}\cdot Pl + \frac{19}{32}\cdot P\cdot x$

$M_2(0) = -\frac{3}{32}Pl$

$M_2\left(\frac{l}{2}\right) = -\frac{3}{32}P\cdot l + \frac{19}{32}P\cdot\frac{l}{2} = \frac{13}{64}Pl$

$\frac{l}{2} \le x_3 \le l$

$M_3(x) = -\frac{3}{32}Pl + \frac{19}{32}\cdot P\cdot x - P\left(x - \frac{l}{2}\right)$

$M_3\left(\frac{l}{2}\right) = \frac{13}{64}Pl$

$M_3(l) = -\frac{3}{32}Pl + \frac{19}{32}\cdot P\cdot l - \frac{Pl}{2} = 0$