

## Maxwell-Mohr method, Verveschagin rule

Graph multiplication method presents most effective way for computation of any displacement (linear, angular, mutual, etc) of bending structures.

The advantage of this method is that the integration procedure according to Maxwell-Mohr integral is replaced by elementary algebraic procedure on two bending moment diagrams in the:

- actual state, and

- unit state.

This method was developed by Russian engineer Verveschagin (1925) and is often referred as the Verveschagin rule.

Let us consider some portion AB which is a part of a bending structure (actual state). The bending stiffness, EI, within of this portion is constant.

The bending moment diagram for this portion in actual state is  $M_i(x)$

The bending moment diagram for the same structure but for elementary loading (unit state) is  $M_i^*(x)$

In general case, a bending moment diagram  $M_i(x)$  in the actual state is bounded by curve, but for unit state it may be bounded by straight line (if the structure is subjected to concentrated elementary (1) force or moment).

$$f = \dots \frac{1}{EI} \sum_i \int M_i(x) \cdot M_i^*(x) dx$$

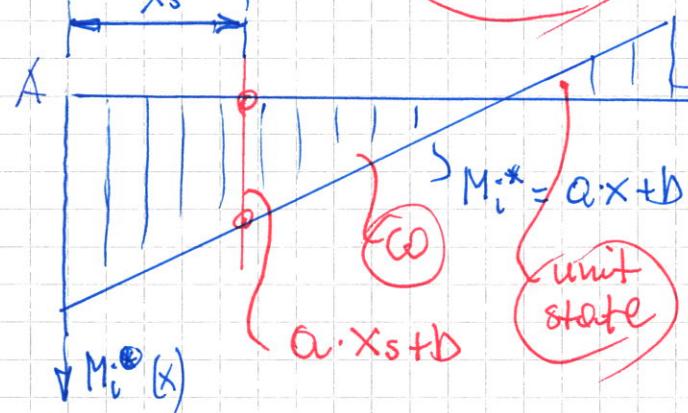
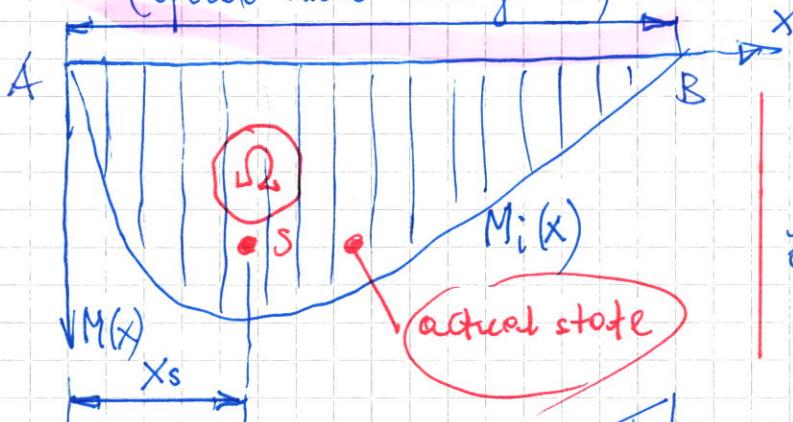
(ii) only for bending (Maxwell-Mohr method)

$$M_i^*(x) = Q \cdot x + b$$

(straight line)

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## Verestchagin procedure (sposób Kleńczyzny)



$$\int_0^L M_i(x) dx = \Delta\Omega$$

$$\int_0^L M_i(x) \cdot x dx = \Delta\Omega \cdot x_s$$

↓

$s$  - centroid of  $M_i(x)$  graph  
 $\Delta\Omega$  - area of bending moment diagram  
 $x_s$  - coordinate of its centroid  
 $\alpha \cdot x_s + b$  - ordinate from the bending moment (unit)

$$\int_0^L M_i(x) \cdot M_i^*(x) dx =$$

$$\int_0^L M_i(x) \cdot (\alpha \cdot x + b) dx =$$

$$\int_0^L M_i(x) \cdot \alpha \cdot x \cdot dx + b \int_0^L M_i(x) dx$$

$$= \alpha \int_0^L M_i(x) \cdot x \cdot dx + b \int_0^L M_i(x) dx =$$

static moment      total area =  $\Delta\Omega$

$$\int_0^L M_i(x) \cdot M_i^*(x) dx =$$

$$\alpha \cdot \Delta\Omega \cdot x_s + b \cdot \Delta\Omega$$

$$\Delta\Omega (\alpha \cdot x_s + b)$$

$$\int_0^L M_i(x) \cdot M_i^*(x) dx = \Delta\Omega (\alpha \cdot x_s + b)$$

procedure  
is called the  
"multiplication" of  
two graphs

$$y(x) = \frac{1}{EI} \left[ \int_0^L M_i(x) \cdot M_i^*(x) dx + \int_0^L M_i(x) \cdot M_i^*(x) dx \right] =$$

$$= \frac{1}{EI} \sum_{i=1}^n \left[ \Delta\Omega_i (\alpha_i x_s + b_i) \right]$$

like  
lii

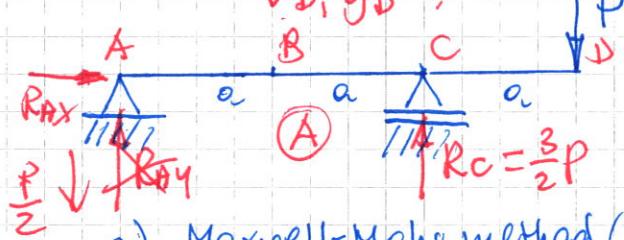
Result of multiplication =  
equals to the area,  $\Delta\Omega$ ,  
of the bending moment diagram  
in actual state multiplied  
by ordinate ( $\alpha \cdot x_s + b$ ) from  
the unit bending moment  $M_i^*$   
which is located under the  
centroid of  $M_i(x)$  diagram

-1 α -

Ex. 1.

$$P, \alpha, EI = \text{const}$$

$$v_D, y_D - ?$$



a) Maxwell-Mohr method (analyt.)

b) M-M, Vereshchagin procedure

(I) Bending m.

$$0 \leq x_1 \leq a$$

$$M_1(x) = -\frac{P}{2} \cdot x$$

$$0 \leq x_2 \leq 2a$$

$$M_2(x) = -\frac{P}{2} \cdot x$$

$$2a \leq x_3 \leq 3a$$

$$M_3(x) = -\frac{P}{2} \cdot x + \frac{3}{2} P(x-2a) = -\frac{P}{2}x + \frac{3}{2}Px - 3Pa = \frac{Px}{2} - 3Pa$$

$$RAy = -\frac{P}{2}$$

$$-\frac{P}{2} + Rc - P = 0 \Rightarrow Rc = \frac{3}{2}P$$

$$2a \leq x_3 \leq 3a$$

$$M_3(x) = -\frac{P}{2} \cdot x + \frac{3}{2} P(x-2a) = -\frac{P}{2}x + \frac{3}{2}Px - 3Pa = \frac{Px}{2} - 3Pa$$

(I) static eqs.

$$\sum P_i x = RAx = 0$$

$$\sum P_i y = RAy - Rc = 0 \Rightarrow RAy = Rc$$

$$\sum M_i c = RAy \cdot 2a - 1 = 0$$

$$RAy = \frac{1}{2a} = Rc$$

(II) Bending moments

$$0 \leq x_1 \leq a$$

$$M_1^*(x) = RAy \cdot x = \frac{x}{2a}$$

$$0 \leq x_2 \leq 2a$$

$$M_2^*(x) = RAy \cdot x = \frac{x}{2a}$$

$$2a \leq x_3 \leq 3a$$

$$M_3^*(x) = RAy \cdot x - Rc(x-2a)$$

$$= \frac{x}{2a} - \frac{1}{2a}(x-2a) =$$

$$= \frac{x}{2a} - \frac{x}{2a} + 1 = 1$$

(I) static eqs.

$$\sum P_i x = RAx = 0$$

$$\sum P_i y = RAy - Rc + 1 = 0$$

$$\sum M_i c = RAy \cdot 2a - 1 \cdot a = 0 \Rightarrow RAy = \frac{1}{2}$$

$$\text{from (2)} \quad \frac{1}{2} - Rc + 1 = 0 \quad Rc = \frac{3}{2}$$

(II) Bending moments

$$0 \leq x_1 \leq a$$

$$M_1^{**}(x) = RAy \cdot x = \frac{x}{2}$$

$$0 \leq x_2 \leq 2a$$

$$M_2^{**}(x) = \frac{x}{2}$$

$$2a \leq x_3 \leq 3a$$

$$M_3^{**}(x) = \frac{x}{2} - \frac{3}{2}(x-2a) = -x + 3a$$

## Metoda M-M (analyt.)

(A) + (B)

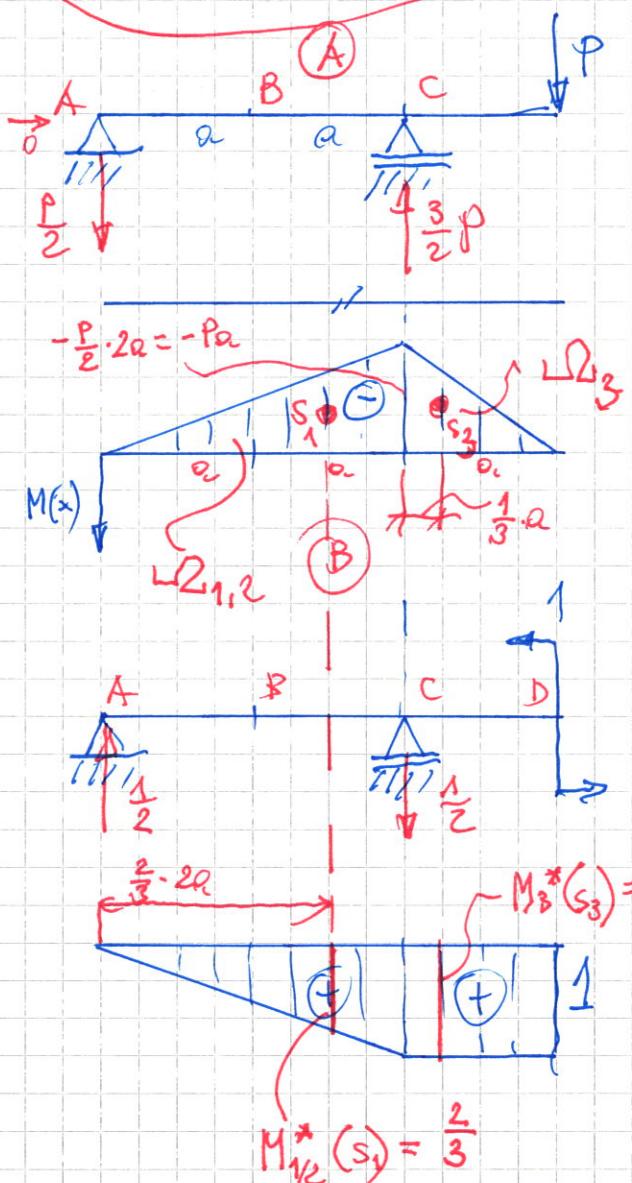
$$\begin{aligned}
 y_B &= \frac{1}{EI} \left[ \int_{0}^a \left( -\frac{P}{2} \cdot x \right) \cdot \frac{x}{2a} dx + \int_{a}^{2a} \left( -\frac{P}{2} \cdot x \right) \cdot \frac{x}{2a} dx + \int_{2a}^{3a} (P \cdot x - 3Pa) \cdot 1 dx \right] = \\
 &= \frac{1}{EI} \left[ \int_{0}^a -\frac{Px^2}{4a} dx + \int_{a}^{2a} -\frac{Px^2}{4a} dx + \int_{2a}^{3a} (Px - 3Pa) dx \right] = \\
 &= \frac{1}{EI} \left[ -\frac{Px^3}{12a} \Big|_0^a - \frac{Px^3}{12a} \Big|_a^{2a} + \left( \frac{Px^2}{2} - 3Pa \cdot x \right) \Big|_{2a}^{3a} \right] = \\
 &= \frac{1}{EI} \left[ -\frac{Pa^3}{12a} - \frac{P}{12a} (8a^3 - a^3) + \left( \frac{P}{2} 9a^2 - 3Pa \cdot 3a - \frac{P}{2} 4a^2 + 3Pa \cdot 2a \right) \right] = \\
 &= \frac{1}{EI} \left[ -\frac{Pa^2}{12} - \cancel{\frac{2}{8} \frac{Pa^2}{12}} - \frac{7}{12} Pa^2 + \left( \frac{9Pa^2}{2} - 9Pa^2 - 2 \cancel{\frac{Pa^2}{4}} + 6Pa^2 \right) \right] = \\
 &= \frac{1}{EI} \left[ -\frac{2}{3} Pa^2 - \frac{Pa^2}{2} \right] = -\frac{7Pa^2}{6EI}
 \end{aligned}$$

(A) + (C)

$$\begin{aligned}
 y_C &= \frac{1}{EI} \left[ \int_{0}^a \left( -\frac{P}{2} \cdot x \right) \frac{x}{2} dx + \int_{a}^{2a} \left( -\frac{P}{2} \cdot x \right) \frac{x}{2} dx + \int_{2a}^{3a} (P \cdot x - 3Pa) \cdot 1 dx \right] = \\
 &= \frac{1}{EI} \left[ -\frac{Px^2}{4} dx + \int_{a}^{2a} -\frac{Px^2}{4} dx + \int_{2a}^{3a} (Px - 3Pa) dx \right] = \\
 &= \frac{1}{EI} \left[ -\frac{Px^3}{12} \Big|_0^a - \frac{Px^3}{12} \Big|_a^{2a} + \left( \frac{Px^2}{2} - 3Pa \cdot x \right) \Big|_{2a}^{3a} \right] = \\
 &= \frac{1}{EI} \left[ -\frac{Pa^3}{12} - \frac{P}{12} (8a^3 - a^3) + \left( \frac{P}{2} 9a^2 - 3Pa \cdot 3a - \frac{P}{2} 4a^2 + 3Pa \cdot 2a \right) \right] = \\
 &= \frac{1}{EI} \left[ -\frac{Pa^3}{12} - \frac{7Pa^3}{12} + Pa^2 \left( \frac{9}{2} - \right. \right. \\
 y_C &= \frac{1}{EI} \left[ \int_{0}^a \left( -\frac{P}{2} \cdot x \right) \frac{x}{2} dx + \int_{a}^{2a} \left( -\frac{P}{2} \cdot x \right) \frac{x}{2} dx + \int_{2a}^{3a} (P \cdot x - 3Pa) (-x + 3a) dx \right] = \\
 &= \frac{1}{EI} \left[ -\frac{Px^2}{4} dx + \int_{a}^{2a} -\frac{Px^2}{4} dx + \int_{2a}^{3a} (-Px^2 + 3Px \cdot a + 3Pa \cdot x - 9Pa^2) dx \right] =
 \end{aligned}$$

$$\begin{aligned}
 y_C &= \frac{P}{EI} \left[ -\frac{Px^3}{12} \Big|_0^a - \frac{Px^3}{12} \Big|_a^{2a} + \left( -\frac{Px^3}{3} + 3Pa \frac{x^2}{2} + 3Pa \frac{x^2}{2} - 9Pa^2 x \right) \Big|_{2a}^{3a} \right] = \\
 &= \frac{1}{EI} \left[ -\frac{Pa^3}{12} - \frac{P}{12} (8a^3 - a^3) + \left( -\frac{P}{3} \cdot 27a^3 + \frac{3Pa}{2} \cdot 9a^2 + \frac{3Pa}{2} \cdot 9a^2 - 9Pa^2 \cdot 3a \right. \right. \\
 &\quad \left. \left. + \frac{P}{3} 8a^3 - \frac{3Pa}{2} 4a^2 - \frac{3Pa}{2} 4a^2 + 9Pa^2 \cdot 2a \right) \right] = \\
 &= \frac{1}{EI} \left[ -\frac{2}{3} Pa^3 + Pa^3 \left( -9 + \frac{27}{2} + \frac{27}{2} - 27 + \frac{8}{3} - 6 - 6 + 18 \right) \right] = \\
 &= \frac{1}{EI} \left( -\frac{2}{3} Pa^3 + 3 Pa^3 + \frac{8}{3} Pa^2 \right) = - \frac{Pa^3}{EI}
 \end{aligned}$$

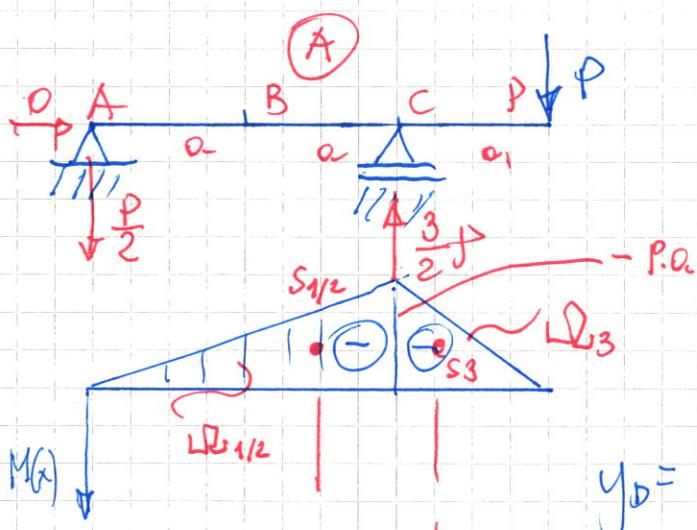
## V. procedure



$$\begin{aligned}
 V_D &= \frac{1}{EI} \left[ \Delta Q_{1/e} \cdot \frac{2}{3} + \Delta Q_3 \cdot 1 \right] = \\
 &= \frac{1}{EI} \left[ \left( -\frac{Pa^2}{2} \right) \cdot \frac{2}{3} + \left( -\frac{Pa^2}{2} \right) \cdot 1 \right] = \frac{-Pa^2}{EI} \left( \frac{2}{3} + \frac{1}{2} \right) = \\
 &= -\frac{7}{6} \frac{Pa^2}{EI}
 \end{aligned}$$

$$\underline{\Omega}_{1/2} = \frac{1}{2} \cdot 2a \cdot (-p.a) = -pa^2$$

$$q_3 = \frac{1}{2} q \cdot (-p) = -\frac{pq^2}{2}$$

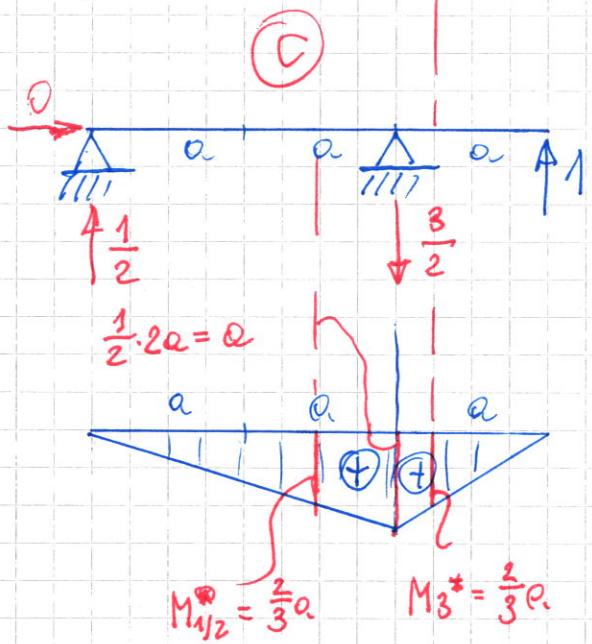


$$y_D = \frac{1}{EI} \left[ \Delta_{1/2} \cdot \frac{2}{3}Q + \Delta_3 \cdot \frac{2}{3}Q \right] =$$

$$y_D = \frac{1}{EI} \left[ (-Pa^2) \cdot \frac{2}{3}Q + \left(-\frac{Pa^2}{2}\right) \cdot \frac{2}{3}Q \right]$$

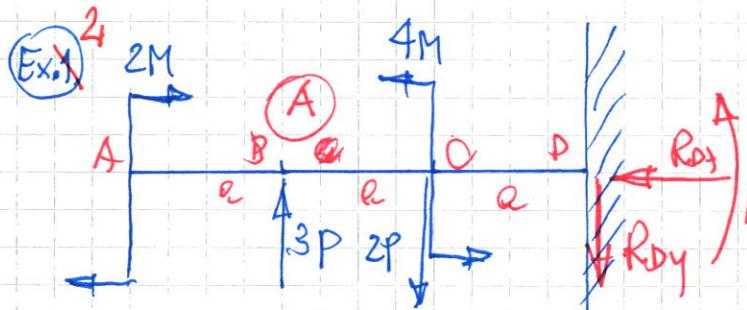
$$y_D = \frac{-Pa^2}{EI} \left( \frac{2}{3}Q + \frac{1}{2} \cdot \frac{2}{3}Q \right) =$$

$$y_D = \frac{-Pa^2}{EI} \cdot \frac{2}{3}Q \cdot \frac{3}{8}a \cdot \frac{3}{8} = -\frac{Pa^3}{EI}$$



$$M_{1/2} = \frac{2}{3}Q$$

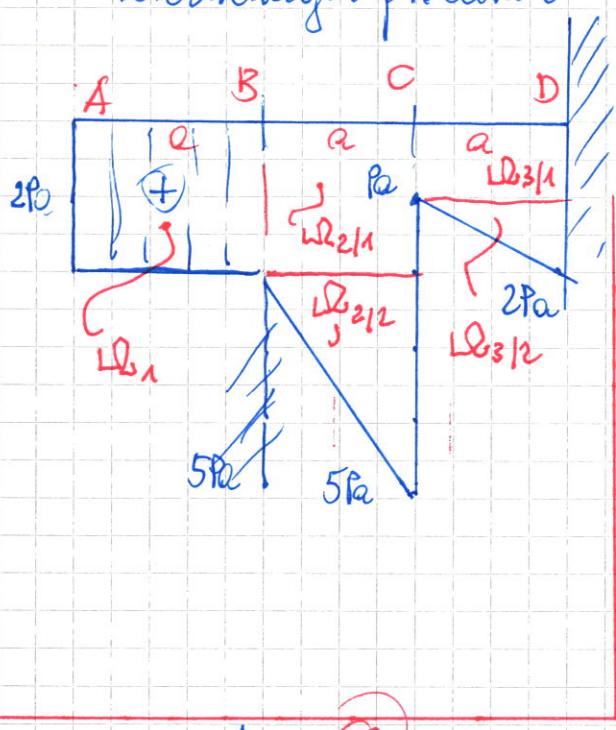
$$M_3 = \frac{2}{3}Q$$



$$P, \alpha, M = Pa, EI = \text{const}$$

$v_B, v_C - ?$

Vereshchagin procedure



I static eqc.

$$\textcircled{1} \sum P_{ix} = -R_{dx} = 0$$

$$\textcircled{2} \sum P_{iy} = 3P - 2P - R_{dy} = 0$$

$$P - R_{dy} = 0 \Rightarrow R_{dy} = P$$

$$\textcircled{3} \sum M_{ib} = 2M + 3P \cdot 2a - 2P \cdot a - 4M - M_D = 0$$

$$2Pa + 6Pa - 2Pa - 4Pa - M_D = 0$$

$$M_D = 2Pa$$

II bending moment

$$0 \leq x_1 \leq a$$

$$M_1(x) = 2M = 2Pa$$

$$a \leq x_2 \leq 2a$$

$$M_2(x) = 2M + 3P(x-a)$$

$$M_2(a) = 2Pa + 3P(a-a) = 2Pa$$

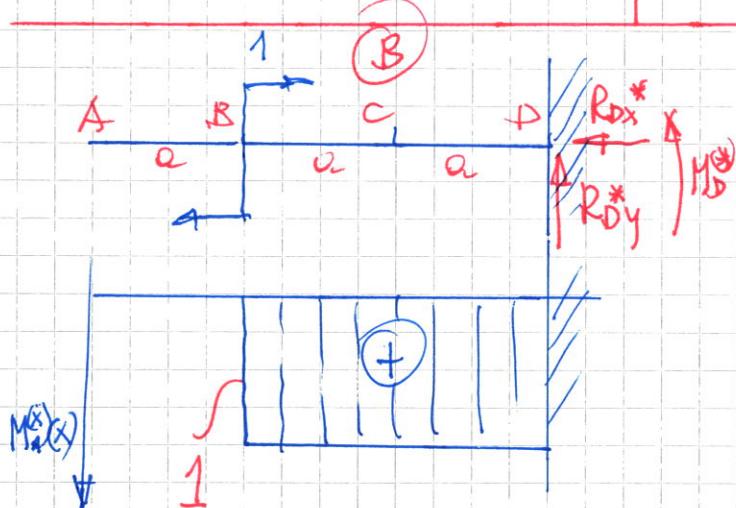
$$M_2(2a) = 2Pa + 3P \cdot a = 5Pa$$

$$2a \leq x_3 \leq 3a$$

$$M_3(2a) = 5Pa - 4Pa = Pa$$

$$M_3(3a) = 2M + 3P \cdot 2a - 2P \cdot a - 4Pa =$$

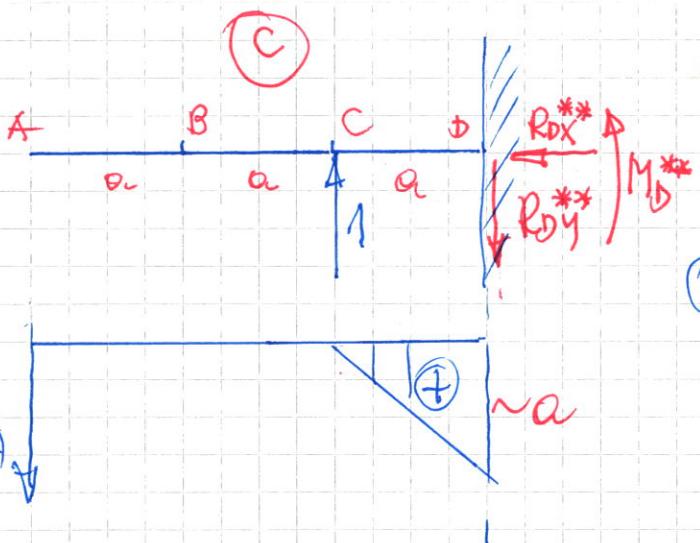
$$2Pa + 6Pa - 2Pa - 4Pa = 2Pe$$



$$M_1^*(x) = 0$$

$$M_2^*(x) = 1$$

$$M_3^*(x) = 1$$



I) static eqs.

$$\textcircled{1} \sum F_{ix} = R_{Dx} = 0$$

$$\textcircled{2} \sum F_{iy} = 1 - R_{Dy}^{**} = 0 \Rightarrow R_{Dy}^{**} = 1$$

$$\textcircled{3} \sum M_D = 1 \cdot a - M_D^{**} = 0 \Rightarrow M_D^{**} = 1/a$$

$$= a$$

$$\Delta_{11} = 2Pa \cdot a = 2Pa^2$$

$$\Delta_{21} = 2Pa^2$$

$$\Delta_{22} = \frac{1}{2}a \cdot 3Pa = \frac{3}{2}Pa^2$$

$$\Delta_{31} = Pa^2$$

$$\Delta_{32} = \frac{1}{2}Pa^2$$

(A) + (B)

$$V_B = \frac{1}{EI} [\Delta_{11} \cdot 0 + \Delta_{21} \cdot 1 + \Delta_{22} \cdot 1 + \Delta_{31} \cdot 1 + \Delta_{32} \cdot 1]$$

$$V_B = \frac{1}{EI} [2Pa^2 \cdot 1 + \frac{3}{2}Pa^2 \cdot 1 + Pa^2 \cdot 1 + \frac{1}{2}Pa^2 \cdot 1] = \frac{5Pa^2}{EI}$$

$$y_C = \frac{1}{EI} [0 + 0 + \Delta_{31} \cdot \frac{a}{2} + \Delta_{32} \cdot \frac{2}{3}a] =$$

$$y_C = \frac{1}{EI} \left[ Pa^2 \cdot \frac{a}{2} + \frac{Pa^2}{2} \cdot \frac{2}{3}a \right] = \frac{1}{EI} \left[ \frac{Pa^3}{2} + \frac{Pa^3}{3} \right]$$

$$y_C = \frac{5}{6EI} Pa^3$$