Thic-walled cylinder subjected to internal and external pressures. Lame's problem

(Rura grubościenna. Zagadnienie Lamego)

Consider a cylinder of inner radius a and outer radius b. Let the cylinder be subjected to an internal pressure p_a and the external pressure p_b

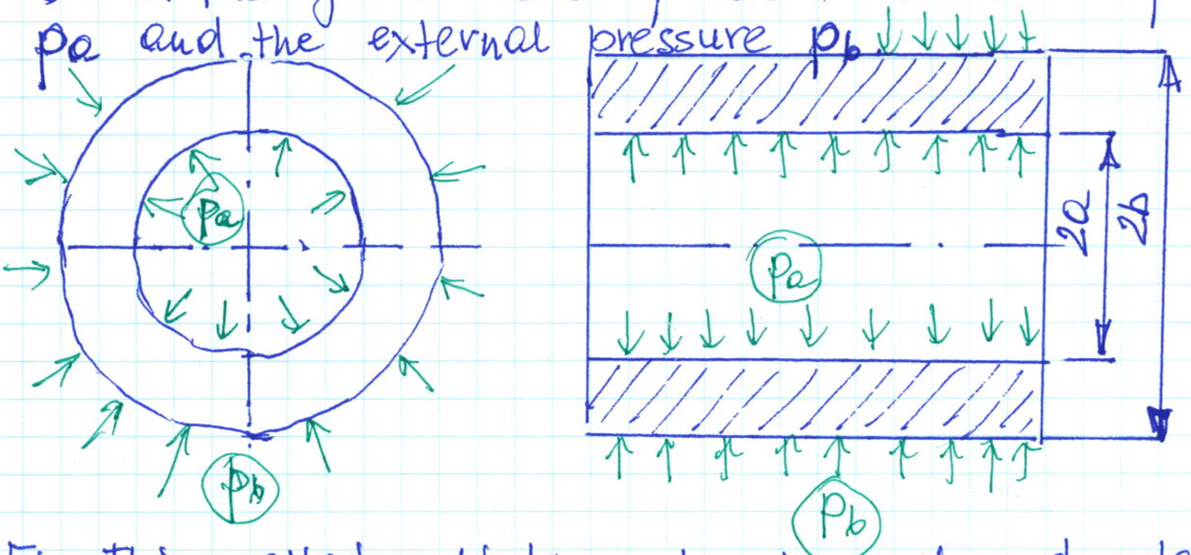
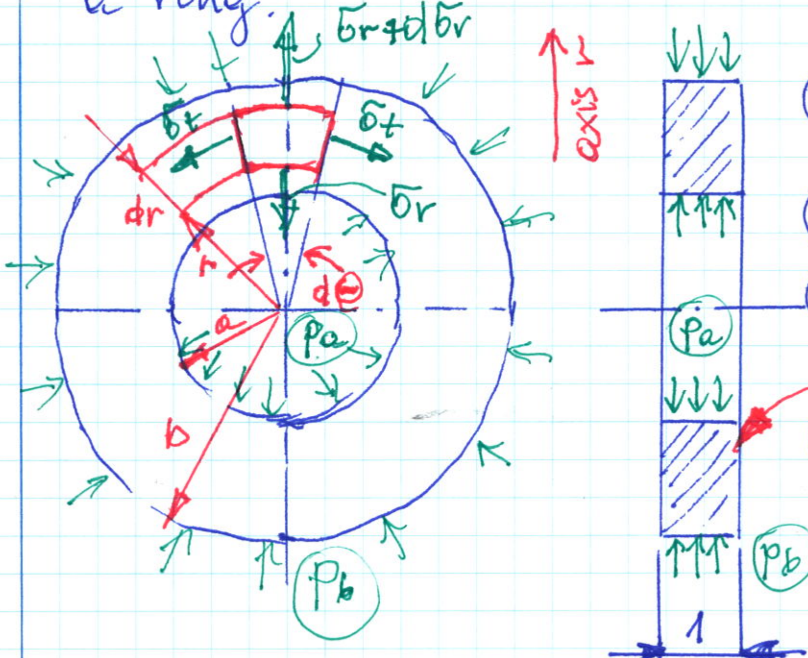


Fig. Thic-walled cylinder under internal and external pressures

We can imagine this problem as an axisymmetric state of stress in a flat disc in the shape of a ring.



Assumptions:

1. the thicknes of the disc $= 1$
2. Hooke's law applies
3. the flat face limiting the disc has no loads
4. for $r = a$; $\sigma_r = -p_a$
   for $r = b$; $\sigma_r = -p_b$
5. the effect of gravity was neglected
6. due to the symmetry of the system we have only one static (equilibrium) equation $\sum P_i^r = 0$

(for red /selected element)

$$(\sigma_r + d\sigma_r)(r + dr)\cdot d\Theta - \sigma_r \cdot r \cdot d\Theta - 2\sigma_t \cdot dr \cdot \sin\frac{d\Theta}{2} = 0 \;!$$

when Θ is small

$$\sin\frac{d\Theta}{2} \cong \frac{d\Theta}{2}$$

then

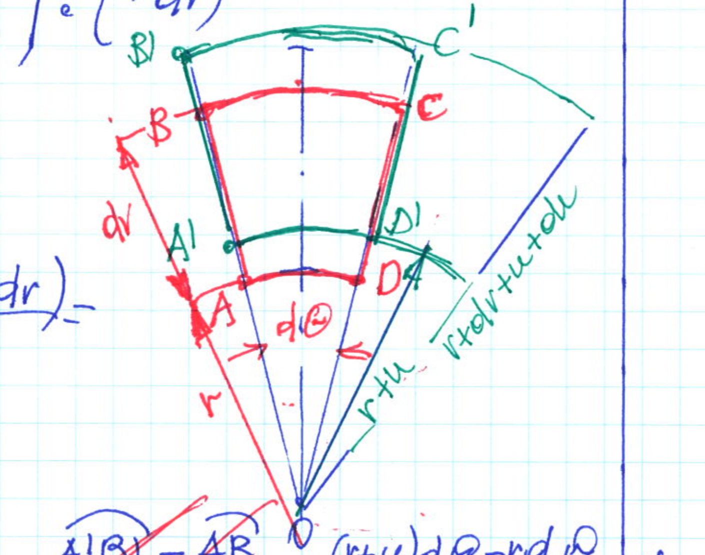
$$(\sigma_r + d\sigma_r)\cdot(r+dr)\cdot d\Theta - \sigma_r\cdot r\cdot d\Theta - 2\sigma_t\cdot dr\cdot \frac{d\Theta}{2} = 0$$

$$\cancel{\sigma_r\cdot r\cdot d\Theta} + d\sigma_r\cdot r\cdot d\Theta + \sigma_r\, dr\cdot d\Theta + \underbrace{d\sigma_r\cdot dr\cdot d\Theta}_{\approx 0 \text{ (very small)}} - \cancel{\sigma_r\cdot r\cdot d\Theta} - \sigma_t\cdot dr\cdot d\Theta = 0$$

$$d\sigma_r\cdot r\cdot \cancel{d\Theta} + \sigma_r\cdot dr\cdot \cancel{d\Theta} - \sigma_t\cdot dr\cdot \cancel{d\Theta} = 0 \quad \Big|: d\Theta$$

$$d\sigma_r\cdot r + \sigma_r\cdot dr - \sigma_t\cdot dr = 0 \quad \Big|: (-dr)$$

(1) $$\boxed{\sigma_t - \sigma_r - r\frac{d\sigma_r}{dr} = 0}$$



u - radial displacement

$$\varepsilon_r = \frac{A'B' - AB}{AB} = \frac{(r+dr+u+du) - (r+dr)}{dr} =$$

$$= \frac{\cancel{r} + \cancel{dr} + u + du - \cancel{r} - \cancel{dr} - u}{dr}$$

$$= \frac{(dr+du) - dr}{dr} = \frac{du}{dr}$$

$$\varepsilon_t = \frac{\cancel{\overset{\frown}{A'B'} - \overset{\frown}{AB}}}{\cancel{\overset{\frown}{AB}}} = \frac{(r+u)\,d\Theta - r\,d\Theta}{r\cdot d\Theta} = \frac{u}{r}$$

$$\varepsilon_t = \frac{\overset{\frown}{A'D'} - \overset{\frown}{AD}}{\overset{\frown}{AD}}$$

(2) $$\boxed{\varepsilon_r = \frac{du}{dr}, \quad \varepsilon_t = \frac{u}{r}}$$

(3) $$\sigma_r = \frac{E}{1-\nu^2}(\varepsilon_r + \nu\varepsilon_t); \quad \sigma_t = \frac{E}{1-\nu^2}(\varepsilon_t + \nu\varepsilon_r)$$

⇐ Hooke's law for a plane stress state

after substituting expression (2) into equations (3) we get

(4) $$\sigma_r = \frac{E}{1-\nu^2}\left(\frac{du}{dr} + \nu\frac{u}{r}\right); \quad \sigma_t = \frac{E}{1-\nu^2}\left(\frac{u}{r} + \nu\frac{du}{dr}\right)$$

and substituting now (4) into (1) we get

(5) $$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0$$

or otherwise

(6) $$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(r\cdot u)\right] = 0 \Rightarrow$$ after expanding the expression, the same

Solving the differential equation has the form:

(7) $$u = C_1 \cdot r + \frac{C_2}{r} \quad \text{and then}$$

(8) $$\begin{cases} \sigma_r = \dfrac{E}{1-\nu^2}\left[(1+\nu)C_1 - (1-\nu)\dfrac{C_2}{r^2}\right] \\ \sigma_t = \dfrac{E}{1-\nu^2}\left[(1+\nu)C_1 + (1-\nu)\dfrac{C_2}{r^2}\right] \end{cases}$$

and boundary conditions:

$$r = a, \quad \sigma_r = -p_a$$

$$r = b, \quad \sigma_r = -p_b$$

Hence

(9) $$\begin{cases} \dfrac{E}{1-\nu^2}\left[C_1(1+\nu) - C_2(1-\nu)\dfrac{1}{a^2}\right] = -p_a \\ \dfrac{E}{1-\nu^2}\left[C_1(1+\nu) - C_2(1-\nu)\dfrac{1}{b^2}\right] = -p_b \end{cases}$$

whence,

(10) $$\begin{cases} C_1 = \dfrac{1-\nu}{E}\,\dfrac{p_a a^2 - p_b b^2}{b^2-a^2} \\ C_2 = \dfrac{1+\nu}{E}\,\dfrac{a^2b^2}{b^2-a^2}(p_a - p_b) \end{cases}$$

substituting this in (7) and (8) we get

(11)

$$u = \frac{1-\nu}{E}\,\frac{p_a a^2 - p_b b^2}{b^2-a^2}\cdot r + \frac{1+\nu}{E}\,\frac{a^2b^2}{r}\,\frac{p_a - p_b}{b^2-a^2} =$$

$$u = \frac{1}{E(b^2-a^2)}\left[(1-\nu)(p_a a^2 - p_b b^2) + (1+\nu)a^2b^2(p_a-p_b)\frac{1}{r}\right]$$

$$\sigma_r = \frac{a^2p_a - b^2p_b}{b^2-a^2} - \frac{a^2b^2(p_a-p_b)}{(b^2-a^2)\cdot r^2}$$

$$\sigma_t = \frac{a^2p_a - b^2p_b}{b^2-a^2} + \frac{a^2b^2(p_a-p_b)}{(b^2-a^2)\cdot r^2}$$

It is interesting to observe that the sum $\sigma_r + \sigma_t$ is constant through the thickness of the wall of the cylinder, i.e. independent of r.

(12) $$\sigma_r + \sigma_t = \frac{2E}{1-\nu} C_1 = 2\,\frac{a^2 p_a - b^2 p_b}{b^2 - a^2}$$

It can also be seen that if

$\sigma_z = 0$! than

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z^{=0} - \nu(\sigma_r + \sigma_t)\right] = \frac{-\nu}{E}(\sigma_r + \sigma_t) =$$

(13) $$\varepsilon_z = -\frac{2\nu}{E}\,\frac{a^2 p_a - b^2 p_b}{b^2 - a^2}$$

Cylinder subjected only to internal pressure

In this case $p_b = 0$ and $p_a = p$, then

$$\sigma_r = \frac{a^2 p_a}{b^2 - a^2} - \frac{a^2 b^2 p_a}{(b^2 - a^2) r^2} = \frac{a^2 p_a}{b^2 - a^2}\left(1 - \frac{b^2}{r^2}\right) =$$

$$= \frac{a^2 p}{b^2 - a^2}\left(1 - \frac{b^2}{r^2}\right)$$

$$\sigma_t = \frac{a^2 p_a}{b^2 - a^2} + \frac{a^2 b^2 p_a}{(b^2 - a^2) r^2} = \frac{a^2 p_a}{b^2 - a^2}\left(1 + \frac{b^2}{r^2}\right)$$

$$= \frac{a^2 p}{b^2 - a^2}\left(1 + \frac{b^2}{r^2}\right)$$

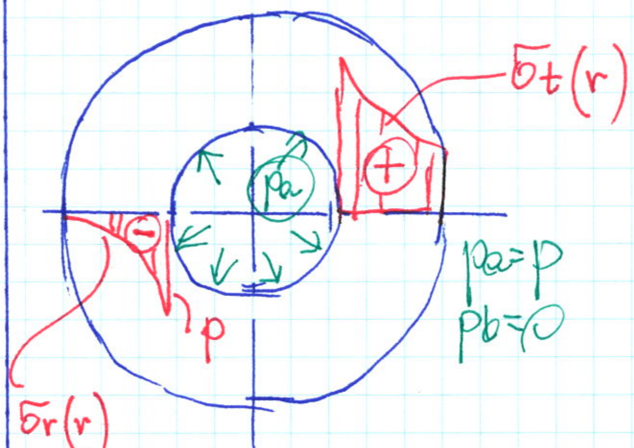
These equations show that σ_r is always a compressive stress and σ_t a tensile stress. Figure below shows the variation of radial (σ_r) and circumferential (σ_t) stresses across the thickness of the cylinder under internal pressure. The circumferential stress is greatest at the inner surface of the cylinder, where

$$\sigma_{t\,max\,(r=a)} = \frac{a^2 p}{b^2 - a^2}\left(1 + \frac{b^2}{a^2}\right) = \cancel{\frac{1}{b^2 a^2}\left(a^2 p +\right.}$$

$$= \frac{\cancel{a^2} p}{b^2 - a^2}\left(\frac{a^2 + b^2}{\cancel{a^2}}\right) = \frac{p\,(a^2 + b^2)}{b^2 - a^2}$$

$$\underset{(r=b)}{\sigma_{t_{min}}} = \frac{a^2 p}{b^2-a^2}\left(1+\frac{b^2}{b^2}\right) = \frac{2a^2 p}{b^2-a^2}$$

Hence, $\sigma_{t_{max}}$ is always greater then the internal pressure and approaches this value as b increases so that it can never be reduced below p_a irrespective of the amount of material added on the outside

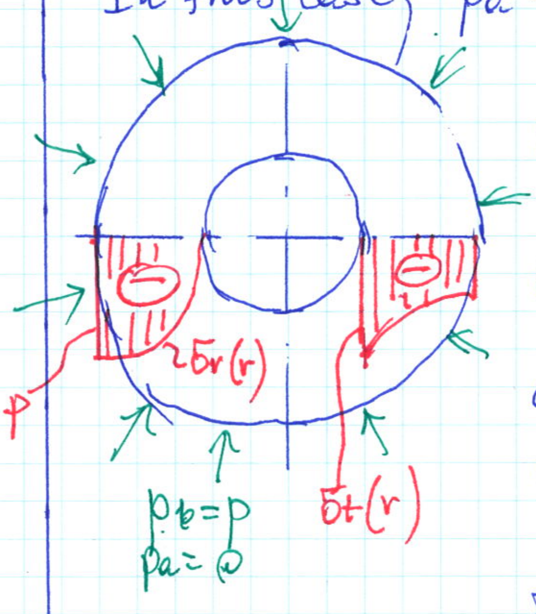


$$\underset{(r=a)}{\sigma_{r\,max}} = \frac{a^2 p}{b^2-a^2}\left(1-\frac{b^2}{a^2}\right) =$$

$$= \frac{a^2 p}{b^2-a^2}\,\frac{(a^2-b^2)}{a^2} = -p$$

Cylinder subjected only to external pressure

In this case, $p_a = 0$ and $p_b = p$, then



$$\sigma_r = \frac{-p b^2}{b^2-a^2}\left(1 - \frac{a^2}{r^2}\right)$$

$$\sigma_t = -\frac{p b^2}{b^2-a^2}\left(1+\frac{a^2}{r^2}\right)$$

and then

$$\underset{(r=a)}{\sigma_{r\,min}} = \frac{-pb^2}{b^2-a^2}\left(1-\frac{a^2}{a^2}\right) = 0\ !$$

$$\underset{(r=b)}{\sigma_{r\,max}} = \frac{-pb^2}{b^2-a^2}\left(1-\frac{a^2}{b^2}\right) = -p$$

$$\underset{(r=a)}{\sigma_{t\,max}} = \frac{-pb^2}{b^2-a^2}\left(1+\frac{a^2}{a^2}\right) = \frac{-2pb^2}{b^2-a^2}$$

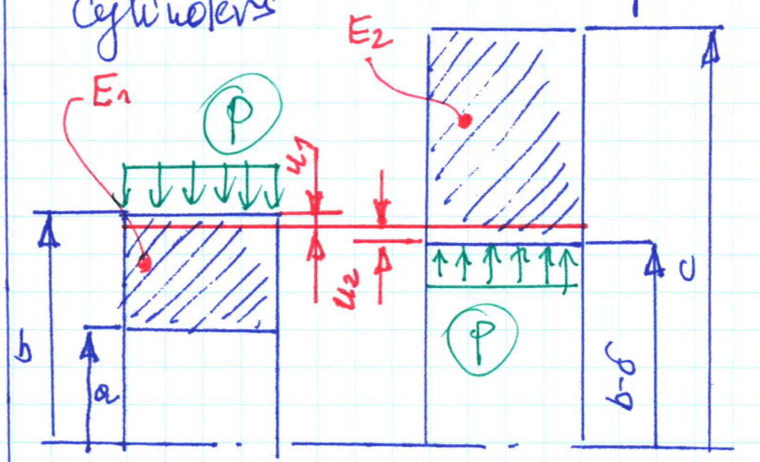
$$\underset{(r=b)}{\sigma_{t\,min}} = \frac{-pb^2}{b^2-a^2}\left(1+\frac{a^2}{b^2}\right)$$

If there is no inner hole, i.e. if $a=0$, the stresses are uniformly distributed in the cylinder with $\sigma_r=\sigma_t=-p$ or more general

$$\sigma_r=\sigma_t=\frac{-p\,b^2}{b^2-\underset{\substack{\parallel \\ 0}}{a^2}}=-p \quad \left(\text{i.e. independent of } r\right)$$

Press-fit connections of two thick-walled cylinders

The problem which will be considered now involves two cylinders made (generally) of two different materials and fitted one inside the other. Before assembling, the inner cylinder has an internal radius a and ~~the~~ an external radius b. The internal radius of ~~of~~ the outer cylinder is less than b by δ, i.e. internal radius is $b-\delta$. External radius of outer cylinder is c. If the inner cylinder cooled and the outer cylinder is heated, then the two cylinders can be assembled, one fitting inside the other. When the cylinders come to room temperature, a shrink fit is obtained. The problem ~~is~~ in determining the contact pressure p between the two cylinders



The contact pressure p acting on the outer surface of the inner cylinder reduces its outer radius by u_1. On the other hand, the other hand, the same contact pressure increases the inner radius of the outer cylinder by u_2

$$|u_1|+|u_2|=\delta$$

The sum of modules of these two quantities must be equal to δ (shrink) because

$$\delta = u_2 - u_1 = |u_2| + |u_1|$$

$$u_1 = \frac{-bp}{E_1}\left(\frac{a^2+b^2}{b^2-a^2} - \nu_1\right)$$

$$u_2 = \frac{bp}{E_2}\left(\frac{b^2+c^2}{c^2-b^2} + \nu_2\right)$$

if (for simplicity) $E_1 = E_2 = E$ and $\nu_1 = \nu_2 = \nu$

$$\delta = u_2 - u_1 = \frac{bp}{E}\left[\left(\frac{b^2+c^2}{c^2-b^2} + \nu\right) + \left(\frac{a^2+b^2}{b^2-a^2} - \nu\right)\right] =$$

$$= \frac{b \cdot p}{E}\left[\frac{b^2+c^2}{c^2-b^2} + \frac{a^2+b^2}{b^2-a^2}\right] = \frac{b \cdot p}{E}\,\frac{(b^2+c^2)(b^2-a^2)+(a^2+b^2)(c^2-b^2)}{(b^2-a^2)(c^2-b^2)}$$

$$\delta = \frac{b \cdot p}{E}\left(\frac{\cancel{b^4} + b^2c^2 - a^2b^2 \cancel{- a^2c^2} \cancel{+ a^2c^2} + b^2c^2 - a^2b^2 \cancel{- b^4}}{(b^2-a^2)(c^2-b^2)}\right)$$

$$= \frac{b \cdot p}{E}\,\frac{2b^2c^2 - 2a^2b^2}{(b^2-a^2)(c^2-b^2)} = \frac{b \cdot p\, 2 \cdot b^2}{E}\,\frac{(c^2-a^2)}{(b^2-a^2)(c^2-b^2)}$$

$$\delta = \frac{2pb^3\,(c^2-a^2)}{E\,(b^2-a^2)(c^2-b^2)} \Rightarrow p$$

$$\boxed{p = \frac{E\delta}{2b^3}\,\frac{(b^2-a^2)(c^2-b^2)}{(c^2-a^2)}}$$

When the outer cylinder is placed on the full shaft, then $a = 0$ and

$$\boxed{p = \frac{E \cdot \delta}{2b^3}\,\frac{b^2(c^2-b^2)}{c^2} = \frac{E\delta\,(c^2-b^2)}{2 \cdot b \cdot c^2}}$$