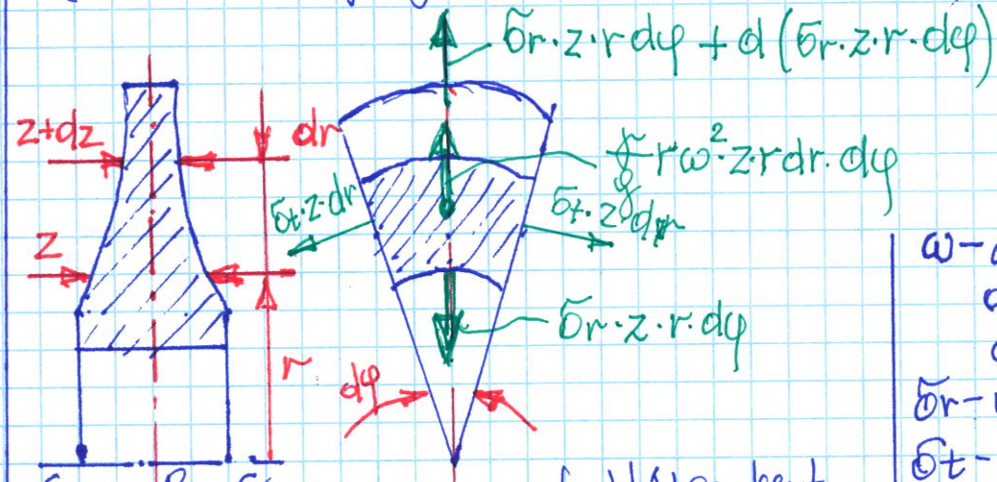
Disks of variable thickness

(Krążek wirujący o zmiennej grubości)



ω – angular velocity of the rotating disc

σ_r – radial stress

σ_t – circumferential stress

γ – specific gravity

g – earth acceleration

$\rho = \frac{\gamma}{g}$ = specific density of disc material

$\frac{\gamma}{g} \cdot r \cdot \omega^2 \cdot z \cdot r \cdot dr \cdot d\varphi$ – inertia force

z – thickness where $z = f(r)$

dm – element of mass

Apply the principle of d'Alambert

The equation of equilibrium

(due to the symmetry of the system, we have only one static equation $\sum P_i^r = 0$)

$$dm = \frac{\gamma}{g} \cdot (r \cdot d\varphi) \cdot z \cdot dr = \frac{\gamma}{g} \cdot z \cdot r\, dr \cdot d\varphi$$

inertia force for element of mass is

$$r \cdot \omega^2 \cdot dm = \frac{\gamma}{g} \cdot r\omega^2 \cdot z \cdot r \cdot dr \cdot d\varphi$$

$$\sum P_i^r = \sigma_r \cdot z \cdot r \cdot d\varphi + d(\sigma_r \cdot z \cdot r \cdot d\varphi) - \sigma_r \cdot z \cdot r \cdot d\varphi - 2 \cdot \sigma_t \cdot z \cdot dr \cdot \sin\frac{d\varphi}{2} + \frac{\gamma}{g} r \cdot \omega^2 \cdot z \cdot r\, dr \cdot d\varphi = 0$$

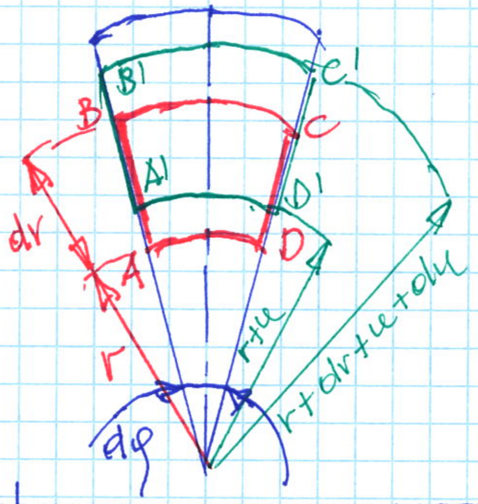
$$d(\sigma_r \cdot z \cdot r \cdot d\varphi) - \sigma_t \cdot z \cdot dr \cdot d\varphi + \frac{\gamma}{g} r^2 \omega^2 z \cdot dr \cdot d\varphi = 0 \quad \Big| : z \cdot dr \cdot d\varphi$$

$$\frac{1}{z}\left[\frac{d(\sigma_r \cdot z \cdot r)}{dr}\right] - \sigma_t + \frac{\gamma}{g} r^2 \omega^2 = 0$$

$$\frac{r}{z} \frac{d(\sigma_r \cdot z)}{dr} + \frac{z}{z}\sigma_r - \sigma_t + \frac{\gamma}{g} r^2 \omega^2 = 0$$

$$\boxed{\frac{r}{z} \frac{d(\sigma_r \cdot z)}{dr} + \sigma_r - \sigma_t + \frac{\gamma}{g} r^2 \omega^2 = 0} \qquad (1)$$

$\sigma_r = \sigma_r(r)$ – ? and $\sigma_t = \sigma_t(r)$ – ?



u - radial displacement

$$\varepsilon_r = \frac{A'B' - AB}{AB} = \frac{r+dr+u+du-(r+dr)}{dr} = \frac{\cancel{r}+\cancel{dr}+u+du-\cancel{r}-\cancel{dr}-u}{dr} = \frac{du}{dr}$$

$$\varepsilon_t = \frac{\widehat{A'D'} - \widehat{AD}}{\widehat{AD}} = \frac{(r+u)d\varphi - r d\varphi}{r \cdot d\varphi} = \frac{u}{r}$$

$$\boxed{\varepsilon_r = \frac{du}{dr}, \quad \varepsilon_t = \frac{u}{r}} \qquad (2)$$

$$\boxed{\begin{aligned} \sigma_r &= \frac{E}{1-\nu^2}(\varepsilon_r + \nu\varepsilon_t) \\ \sigma_t &= \frac{E}{1-\nu^2}(\varepsilon_t + \nu\varepsilon_r) \end{aligned}} \qquad (3)$$

← Hooke's law for a plane stress state

after substituting expression (2) into equations (3) we get

$$\boxed{\sigma_r = \frac{E}{1-\nu^2}\left(\frac{du}{dr} + \nu\frac{u}{r}\right); \quad \sigma_t = \frac{E}{1-\nu^2}\left(\frac{u}{r} + \nu\frac{du}{dr}\right)} \qquad (4)$$

and in equation (1), a derivative is also needed

$$\frac{d(\sigma_r \cdot z)}{dr} = z \cdot \frac{d\sigma_r}{dr} + \sigma_r \frac{dz}{dr} \qquad \text{but}$$

$$\frac{d\sigma_r}{dr} = \frac{E}{1-\nu^2}\left(\frac{d^2u}{dr^2} + \frac{\nu}{r^2}\,\frac{\frac{du}{dr}\cdot r - u}{1}\right) = \frac{E}{1-\nu^2}\left[\frac{d^2u}{dr^2} + \frac{\nu}{r}\frac{du}{dr} - \nu \cdot \frac{u}{r^2}\right]$$

and finally

$$\frac{d(\sigma_r \cdot z)}{dr} = \frac{E}{1-\nu^2} \cdot z\left[\frac{d^2u}{dr^2} + \frac{\nu}{r}\frac{du}{dr} - \nu\frac{u}{r^2}\right] + \frac{E}{1-\nu^2}\frac{dz}{dr}\left[\frac{du}{dr} + \nu\frac{u}{r}\right]$$

substituting all of this into equation (1)

$$\frac{r}{z}\frac{E}{1-\nu^2} \cdot z\left[\frac{d^2u}{dr^2} + \frac{\nu}{r}\frac{du}{dr} - \nu\frac{u}{r^2}\right] + \frac{r}{z}\frac{E}{1-\nu^2}\frac{dz}{dr}\left[\frac{du}{dr} + \nu\frac{u}{r}\right] + \sigma_r - \sigma_t + \frac{\gamma}{g} r^2 \omega^2 = 0$$

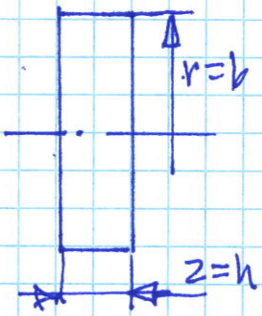
and after transformations

$$\boxed{\frac{d^2u}{dr^2}+\left(\frac{1}{r}+\frac{1}{z}\frac{dz}{dr}\right)\frac{du}{dr}+\frac{1}{r}\left(\frac{\nu}{z}\frac{dz}{dr}-\frac{1}{r}\right)u=A\cdot r} \quad (5)$$

where

$$A=-\frac{1-\nu^2}{E}\,\frac{\gamma}{g}\,\omega^2 \quad (6)$$

Rotating disc of uniform thickness



for $z=h=\text{const}$

equation (5) takes the form

$$\frac{d^2u}{dr^2}+\left(\frac{1}{r}+\frac{1}{h}\cdot 0\right)\frac{du}{dr}-\frac{1}{r^2}u=A\cdot r$$

and finally

$$\boxed{\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}-\frac{u}{r^2}=A\cdot r} \quad (7)$$

this equation can also be written as

$$\boxed{\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(u\cdot r)\right]=A\cdot r} \quad (8)$$

Verification of the correctness of the transformation

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(u\cdot r)\right]=\frac{d}{dr}\left[\frac{1}{r}\left(\frac{du}{dr}\cdot r+u\right)\right]=\frac{d}{dr}\left[\frac{du}{dr}+\frac{u}{r}\right]=$$

$$=\frac{d^2u}{dr^2}+\left(r\frac{du}{dr}-u\right)\frac{1}{r^2}=\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}-\frac{u}{r^2}$$

Now we can integrate an expression (8) twice, namely:

$$\frac{1}{r}\frac{d(u\cdot r)}{dr}=A\frac{r^2}{2}+C_1 \quad \Big|\cdot r$$

$$\frac{d(u\cdot r)}{dr}=A\frac{r^3}{2}+C_1\cdot r \quad \text{and}$$

$$u\cdot r=A\frac{r^4}{8}+C_1\frac{r^2}{2}+C_2$$

$$\boxed{u=A\frac{r^3}{8}+C_1\frac{r}{2}+\frac{C_2}{r}} \quad (9) \qquad C_1, C_2 - ?$$

boundary conditions

(1) $r=0$, $u=0$

(2) $r=b$, $\sigma_r=0$!

① → $C_2 = 0$

② → $$\sigma_r = 0 = \frac{E}{1-\nu^2}\left[A\frac{(3+\nu)b^2}{8} + \frac{1+\nu}{2}\cdot C_1\right]$$

$$C_1 = \frac{-A(3+\nu)b^2\cdot 2}{8(1+\nu)} = -Ab^2\frac{(3+\nu)}{4(1+\nu)}$$

$$A = -\frac{1-\nu^2}{E}\frac{\gamma}{g}\omega^2$$

$$C_1 = \frac{3+\nu}{4(1+\nu)}\frac{1-\nu^2}{E}\frac{\gamma}{g}\omega^2 b^2$$

when inserted to equations for u, σ_r, σ_t

finally

$$\boxed{\begin{aligned} \sigma_r &= \frac{3+\nu}{8}\frac{\gamma\omega^2}{g}\left(b^2 - r^2\right) \\ \sigma_t &= \frac{3+\nu}{8}\frac{\gamma\omega^2}{g}\left(b^2 - \frac{1+3\nu}{3+\nu}\cdot r^2\right) \\ u &= \frac{1-\nu^2}{8E}\frac{\gamma}{g}r\omega^2\left(\frac{3+\nu}{1+\nu}b^2 - r^2\right) \end{aligned}} \qquad (10)$$

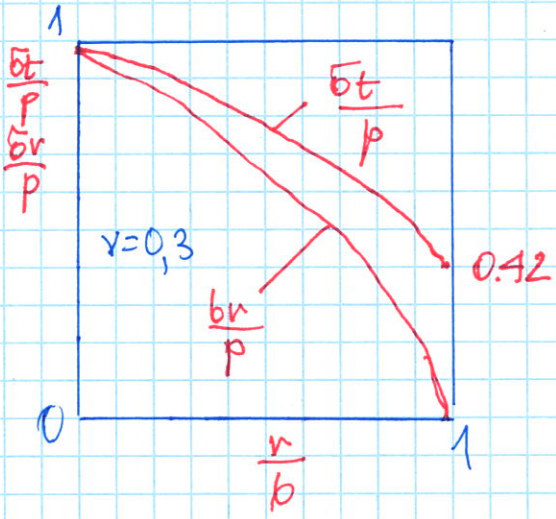
for $r = 0$ $\sigma_r = \sigma_t = p$ (uniform tension)

$$\boxed{\sigma_r = \sigma_t = p = \frac{3+\nu}{8}\frac{\gamma b^2\omega^2}{g}}$$

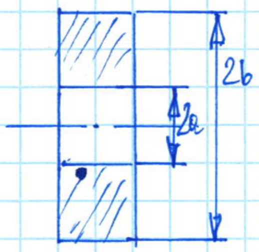
After substituting p for formulas (10)

$$\frac{\sigma_r}{p} = 1 - \left(\frac{r}{b}\right)^2$$

$$\frac{\sigma_t}{p} = 1 - \frac{1+3\nu}{3+\nu}\left(\frac{r}{b}\right)^2$$



Rotating disc with a hole $(r=a)$



$\sigma_r = 0 \quad , r = a$

$\sigma_r = 0 \quad , r = b$

$$\frac{E}{1-\nu^2}\left[A\frac{(3+\nu)a^2}{8} + \frac{1+\nu}{2}C_1 - \frac{1-\nu}{a^2}C_2\right] = 0$$

$$A\frac{(3+\nu)a^2}{8} + \frac{1+\nu}{2}C_1 - \frac{1-\nu}{a^2}C_2 = 0$$

$$C_1 = -\frac{3+\nu}{4(1+\nu)}A\,(a^2+b^2)$$

$$C_2 = -\frac{3+\nu}{8(1-\nu)}A\,a^2b^2$$

and

$$\sigma_r = p\left[1+\left(\frac{a}{b}\right)^2 - \left(\frac{r}{b}\right)^2 - \left(\frac{a}{r}\right)^2\right]$$

$$\sigma_t = p\left[1+\left(\frac{a}{b}\right)^2 - \frac{1+3\nu}{3+\nu}\left(\frac{r}{b}\right)^2 + \left(\frac{a}{r}\right)^2\right]$$

$$u = \frac{p}{E}\left[(1-\nu)(a^2+b^2)r + (1+\nu)\frac{a^2b^2}{r} - \frac{1-\nu^2}{3+\nu}r^3\right]$$

$$p = \frac{3+\nu}{8}\frac{\gamma b^2\omega^2}{g}$$

$$\sigma_t - \sigma_r = 2p\left[\frac{1-\nu}{3+\nu}\left(\frac{r}{b}\right)^2 + \left(\frac{a}{r}\right)^2\right]$$ is always positive $(\sigma_t > \sigma_r)$

$$\underset{(r=a)}{\sigma_{t\,max}} = p\left[1+\left(\frac{a}{b}\right)^2 - \frac{1+3\nu}{3+\nu}\left(\frac{a}{b}\right)^2 + \left(\frac{a}{a}\right)^2\right] = p\left[2+\left(\frac{a}{b}\right)^2\left(1-\frac{1+3\nu}{3+\nu}\right)\right]$$

$$\sigma_{t\,max} = 2p\left[\frac{1-\nu}{3+\nu}\left(\frac{a}{b}\right)^2 + 1\right]$$

if $a \to b$, then $r \to b$ and $\sigma_r \to 0$

$$\sigma_t \to p\left[3 - \frac{1+3\nu}{3+\nu}\right] = \frac{3+\nu}{8}\frac{\gamma b^2\omega^2}{g}\cdot\frac{3(3+\nu)-(1+3\nu)}{3+\nu}$$

$$\sigma_t \Rightarrow \frac{\gamma b^2\omega^2}{g}$$ (thin rotating ring)

if $\frac{a}{r} \Rightarrow 0$, then

$\sigma_{t\,max} \Rightarrow 2p$ (for disc without hole $\sigma_{t\,max} \Rightarrow p$

In the case of "puncture" against an infinitely small hole, a stress ~~conce~~ concentration (2x) occurs.