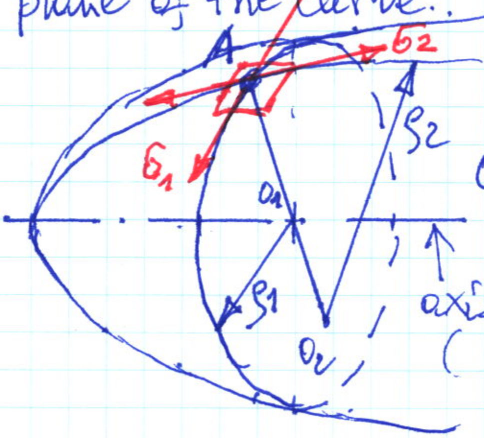
Thin-walled vessels
(zbiorniki cienkościenne)

Thin-walled pressure vessels such as cylindrical, spherical, conical, or toroidal shells subject to internal or external pressure from a gas or a liquid.

The shell of revolution is formed by rotating a plane curve ~~of~~ (the meridian) about an axis lying in the plane of the curve.



$O_1A = \rho_1$ – latitudinal radius (promień równoleżnikowy)

$O_2A = \rho_2$ – meridional radius (promień południkowy)

$\frac{1}{\rho_1}$ – latitudinal curvature

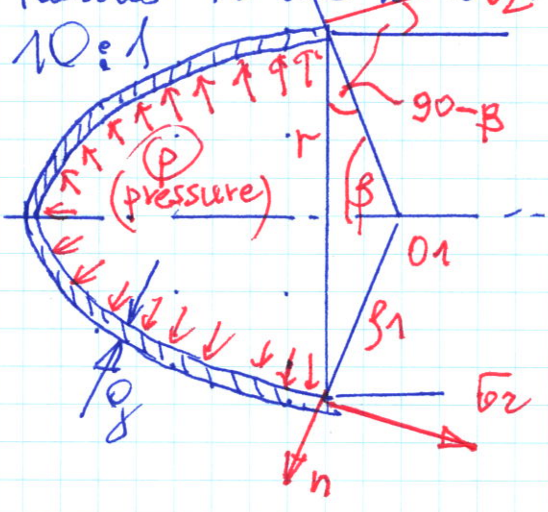
$\frac{1}{\rho_2}$ – meridional curvature

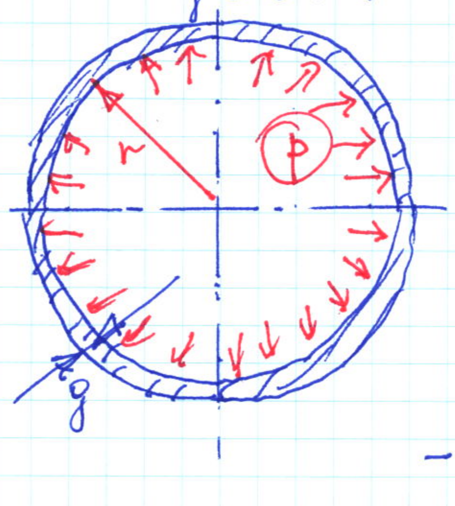
σ_1 – latitudinal stress (hoop or circumferential)

σ_2 – meridional stress

Examples of pressure vessels
- compressed tanks
- pipelines
- water tower
- balloon
- airplanes
- inflatable boats

Thin-walled structures if ratio of the inside radius to the wall thickness is greater than about 10:1





The pressure p acts on the cut-off part of the vessel, giving the force P, where

$$P = p \cdot F \quad \text{, where } F \text{ – projection of the cut-off part of the vessel}$$

$$F = \pi \cdot r^2 \quad \text{, where } r = \rho_1 \cdot \sin\beta$$

and substitute

$$P = p \cdot F = p \cdot \pi r^2 = p \cdot \pi \cdot \rho_1^2 \sin^2\beta$$

The pressure force P is balanced by the force in the vessel wall (F_{wall}), i.e.

$$P = F_{wall} \quad \text{, where}$$

$$\cancel{A_{wall} \cong 2\pi r \cdot g = 2\pi g \cdot \rho_1 \cdot \sin\beta}$$

$$F_{wall} = A_{wall} \cdot \sigma_2 \cdot \sin\beta$$

$$\text{wall section} = A_{wall} = 2\pi r \cdot g = 2\pi g \cdot \rho_1 \cdot \sin\beta$$

and substitute

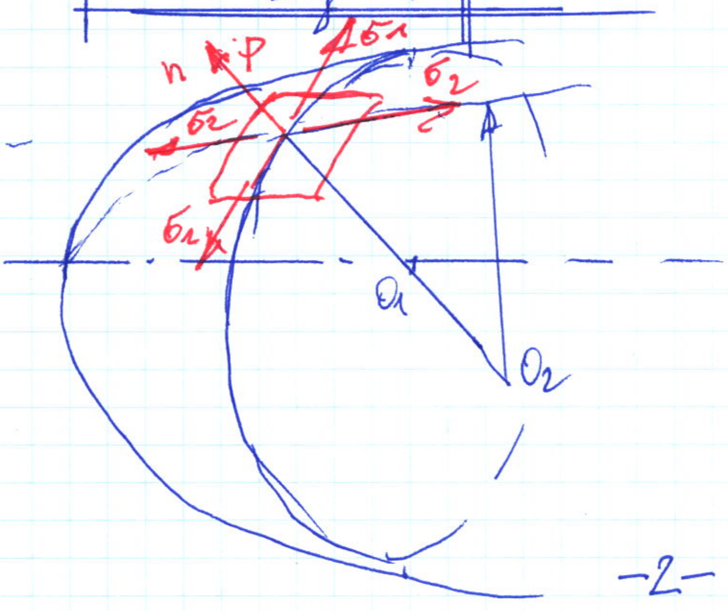
$$F_{wall} = (2\pi g \cdot \rho_1 \cdot \sin\beta) \cdot \sigma_2 \cdot \sin\beta = 2\pi g \rho_1 \sigma_2 \sin^2\beta$$

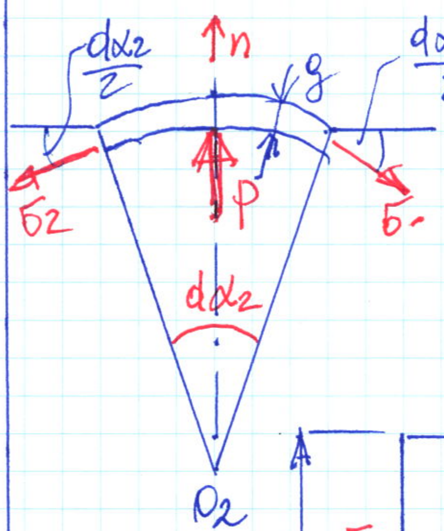
$$P = F_{wall}$$

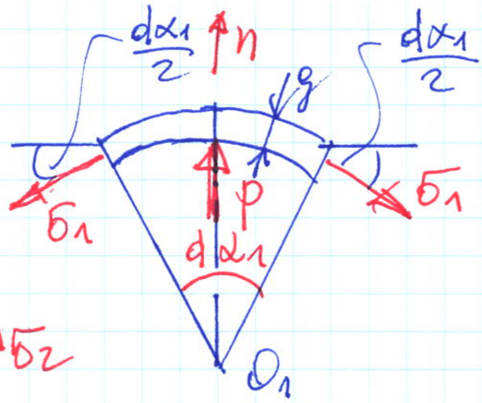
$$p \cdot \cancel{\pi} \cdot \cancel{\rho_1^2} \cdot \cancel{\sin^2\beta} = 2\cancel{\pi} g \cancel{\rho_1} \sigma_2 \cancel{\sin^2}\beta$$

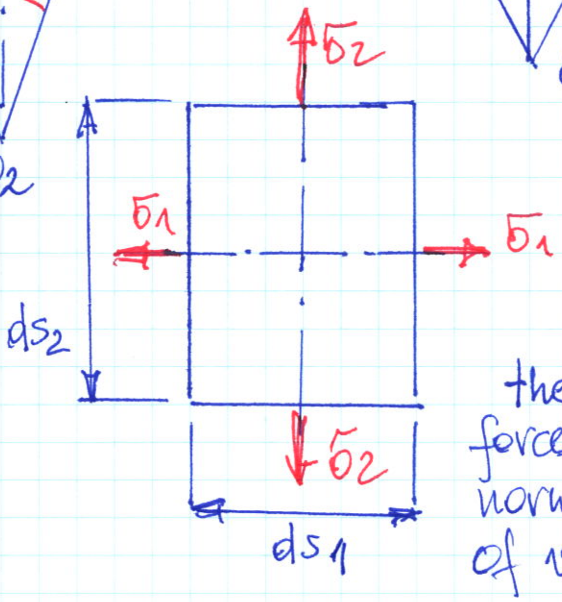
$$\Downarrow$$

$$\boxed{\sigma_2 = \frac{\rho_1}{2g} \cdot p}$$ meridional stress









$$ds_1 = \rho_1 \cdot d\alpha_1$$

$$ds_2 = \rho_2 \cdot d\alpha_2$$

the sum of projections all forces on the direction of normal to the surface of vessel ($\uparrow n$)

$$\sum p_n = p \cdot ds_1 \cdot ds_2 - 2\sigma_1 \cdot g \cdot ds_2 \cdot \sin\frac{d\alpha_1}{2} - 2\sigma_2 \cdot g \cdot ds_1 \cdot \sin\frac{d\alpha_2}{2} = 0$$

for small angles:

$$\sin\frac{d\alpha_1}{2} \simeq \frac{d\alpha_1}{2} \quad \text{and} \quad \sin\frac{d\alpha_2}{2} \simeq \frac{d\alpha_2}{2}$$

$$p \cdot ds_1 \cdot ds_2 - 2\sigma_1 \cdot g \cdot ds_2 \cdot d\alpha_1 - 2\sigma_2 g \cdot ds_1 \cdot \frac{d\alpha_2}{2} = 0$$

$$\frac{p}{g} - \sigma_1 \frac{d\alpha_1}{ds_1} - \sigma_2 \frac{d\alpha_2}{ds_2} = 0 \qquad \Big| : g \cdot ds_1 \cdot ds_2$$

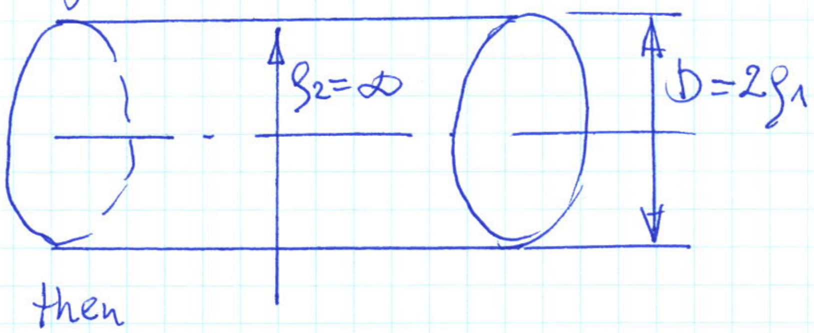
but $\dfrac{d\alpha_1}{ds_1} = \dfrac{1}{\rho_1}$ and $\dfrac{d\alpha_2}{ds_2} = \dfrac{1}{\rho_2}$ then finally

$$\boxed{\frac{\sigma_1}{\rho_1} + \frac{\sigma_2}{\rho_2} = \frac{p}{g}}$$

Laplace formula (basic equation of the shell state)

Cylindrical shells

p, D, g



$$\rho_1=\frac{D}{2}$$

$$\rho_2=\infty$$

so-called boiler's formulas

then

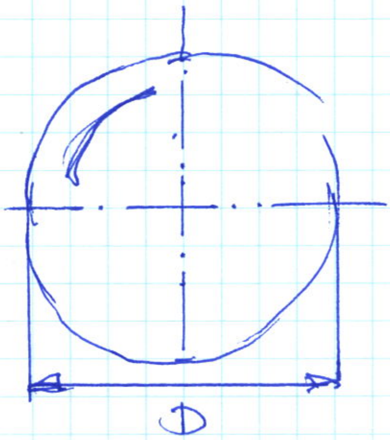
$$\sigma_2=\frac{\rho_1}{2g}\cdot p=\frac{D}{2\cdot 2g}\cdot p=\boxed{\frac{D\cdot p}{4g}}$$

from Laplace's formula

$$\frac{\sigma_1}{\rho_1}+\frac{\sigma_2}{\rho_2}=\frac{p}{g}\Rightarrow\frac{\sigma_1}{\frac{D}{2}}+\overbrace{\cancel{\frac{D\cdot p}{4g\cdot\infty}}}^{=0}=\frac{p}{g}$$

$$\frac{2\sigma_1}{D}=\frac{p}{g}\Rightarrow\boxed{\sigma_1=\frac{Dp}{2g}}$$

Spherical shells (vessels)



p, D, g

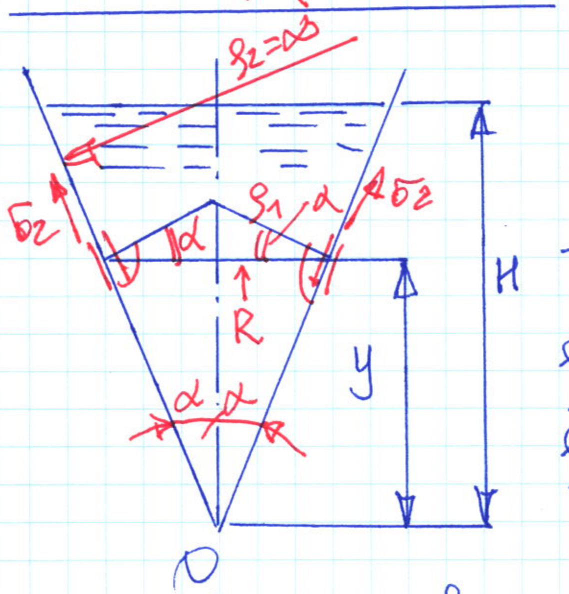
$$\rho_1=\rho_2=\frac{D}{2} \qquad \sigma_1=\sigma_2$$

$$\sigma_1=\sigma_2=\frac{D\cdot p}{4g}$$

$$\sigma_{2\,(cylindrical)}=\sigma_{2\,(spherical)}$$

$$\sigma_1\,(cylindrical)=2\cdot\sigma_1\,(spherical)$$

Conical ~~tank~~ shell



H, α, γ, g

$$\frac{R}{y}=tg\alpha \Rightarrow R=y\cdot tg\alpha$$

$$\frac{R}{\rho_1}=\cos\alpha \Rightarrow \rho_1=\frac{R}{\cos\alpha}$$

$$\rho_1=\frac{R}{\cos\alpha}=\frac{y\cdot tg\alpha}{\cos\alpha}$$

$$\rho_2=\infty$$

At a distance y from the top O, the hydrostatic p_n pressure is

$$p_n=\gamma\,(H-y)$$

but

$$\frac{\sigma_1}{\rho_1}+\frac{\sigma_2}{\infty}=\frac{p_n}{g}$$

$$\frac{\sigma_1\cdot\cos\alpha}{y\cdot tg\alpha}+0=\frac{\gamma(H-y)}{g}\Rightarrow \boxed{\sigma_1=\frac{\gamma\cdot y\,(H-y)\,tg\alpha}{g\cdot\cos\alpha}}$$

the sum of projections of all forces on the direction to the vessel axis is

$$\underset{\text{force in the wall}}{\sigma_2\cdot 2\pi R\cdot g\cdot\cos\alpha}-\underset{\text{force from the hydrostatic pressure}}{p_n\cdot\pi R^2}-\underset{\text{fluid weight for height}=y}{\frac{1}{3}\pi R^2\cdot\gamma\cdot y}=0$$

$$\sigma_2\cdot 2\pi g\cdot\cos\alpha\cdot y\cdot tg\alpha-\gamma(H-y)\cdot\pi y^2\cdot tg^2\alpha-\frac{1}{3}\pi\gamma\cdot y\, y^2 tg^2\alpha=0$$

$$\sigma_2\cdot 2\cancel{\pi}g\cancel{y}\cdot\cos\alpha\cdot\cancel{tg\alpha}-\gamma(H-y)\cancel{\pi}y^{\cancel{2}}tg^{\cancel{2}}\alpha-\frac{1}{3}\cancel{\pi}\gamma y^{\cancel{3}\,2}tg^{\cancel{2}}\alpha=0 \quad \Big|:\pi y\,tg\alpha$$

$$\sigma_2\cdot 2g\cdot\cos\alpha=\gamma(H-y)\cdot y\cdot tg\alpha+\frac{1}{3}\gamma y^2 tg\alpha$$

$$\sigma_2\cdot 2g\cdot\cos\alpha=\gamma y\,tg\alpha\left(H-y+\frac{1}{3}y\right)=\gamma\cdot y\cdot tg\alpha\left(H-\frac{2}{3}y\right)$$

$$\boxed{\sigma_2=\frac{\gamma\cdot y\left(H-\frac{2}{3}y\right)\cdot tg\alpha}{2g\cdot\cos\alpha}}$$

$\sigma_{2max} = ?$

$$\frac{d\sigma_2(y)}{dy} = 0 \Rightarrow H - \frac{4}{3}y = 0 \Rightarrow y = \frac{3}{4}H$$

$$\sigma_{2max} = \frac{\gamma \cdot tg\alpha}{2 \cdot g \cdot \cos\alpha} \frac{3}{4}H\left(H - \frac{2}{\cancel{3}} \cdot \frac{\cancel{3}}{4}H\right) =$$

$$\frac{\gamma \cdot tg\alpha}{2g\cos\alpha} \frac{3}{4}H \frac{H}{2} = \frac{3}{16} \frac{H^2 \cdot \gamma \cdot tg\alpha}{g \cdot \cos\alpha} = \sigma_{2max}$$

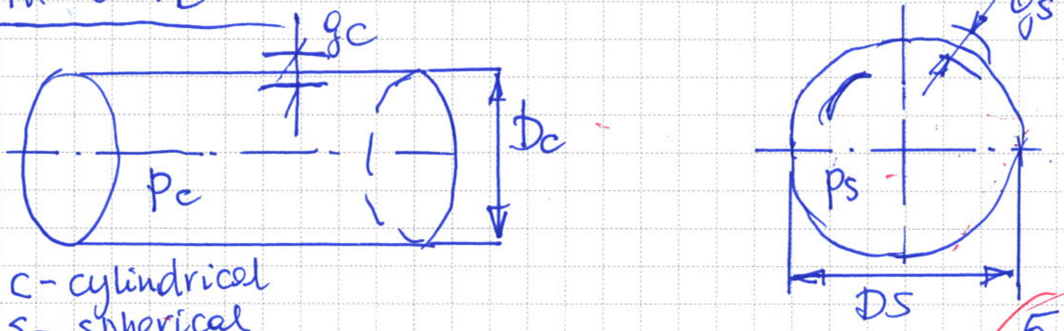
$\sigma_{1max} = ?$

$$\frac{d\sigma_1(y)}{dy} = 0 \Rightarrow H - 2y = 0 \Rightarrow y = \frac{H}{2}$$

$$\sigma_{1max} = \frac{\gamma \cdot \frac{H}{2}\left(H - \frac{H}{2}\right) \cdot tg\alpha}{g \cdot \cos\alpha} = \frac{\gamma \cdot tg\alpha \frac{H^2}{4}}{g \cdot \cos\alpha}$$

$$\sigma_{1max} = \frac{H^2 \cdot \gamma \cdot tg\alpha}{4 \cdot g \cdot \cos\alpha}$$

Homework



c - cylindrical
s - spherical

$D_c = D$, $D_s = 1{,}5D$, $g_c = g$, $g_s = 1{,}2g$
$p_c = 2p$, $p_s = 3p$

$\frac{\sigma_{1c}}{\sigma_{1s}} = ?$

$\frac{\sigma_{2c}}{\sigma_{2s}} = ?$