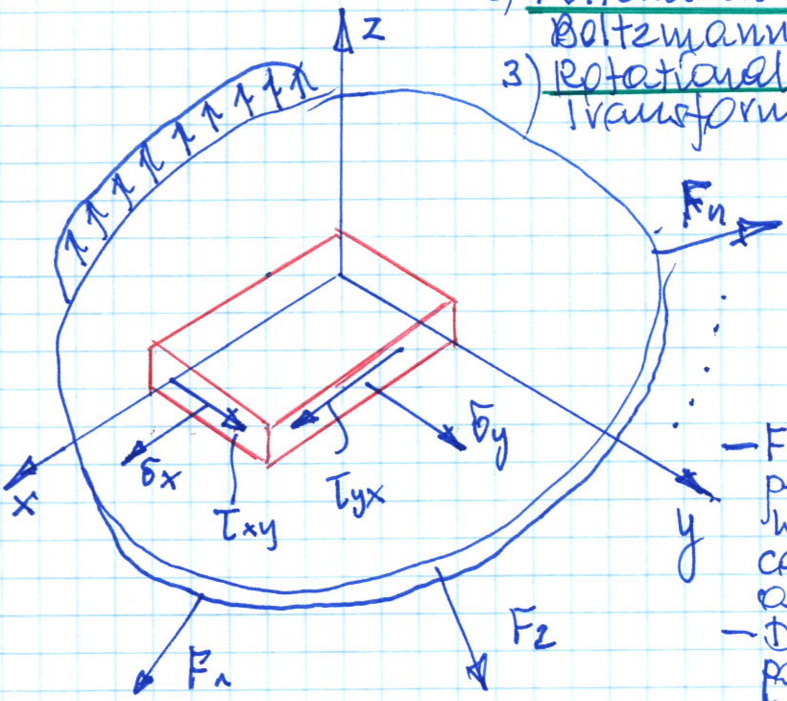
1) Plane stress analysis
Analiza płaskiego stanu naprężenia

2) Boltzmann axiom. Postulat Boltzmanna

3) Rotational transformation
Transformacja obrotowa

4) Principal stresses
Naprężenia główne

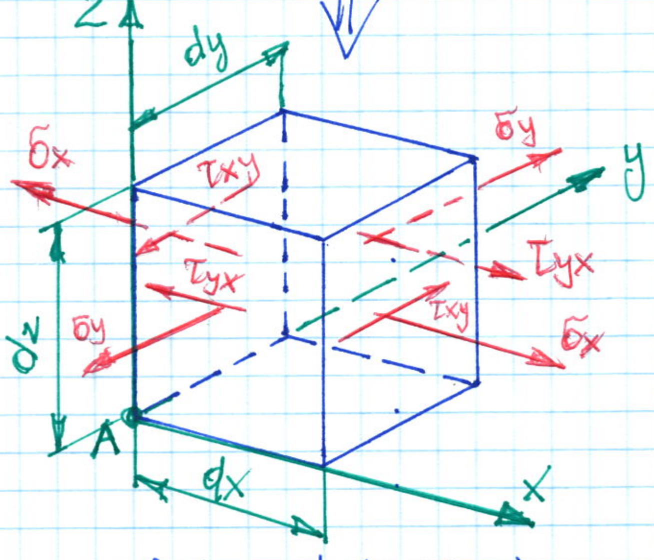
5) Mohr's circle
Koło Mohra



- For any point A, three perpendicular sections were drawn. One section coincides with the plane of stress state
- Dla dowolnego punktu A poprowadzono trzy dowolne prostopadłe przekroje. Jeden przekrój pokrywa się z płaszczyzną stanu naprężenia
- Infinitely close to point A, a second three of mutually perpendicular sections are drawn, respectively parallel to the previous three.
- Nieskończenie blisko punktu A poprowadzono drugą trójkę wzajemnie ⊥ przekrojów odpowiednio równoległych do poprzedniej trójki
- A cuboid dx, dy, dz was created
- Otrzymano prostopadłościan

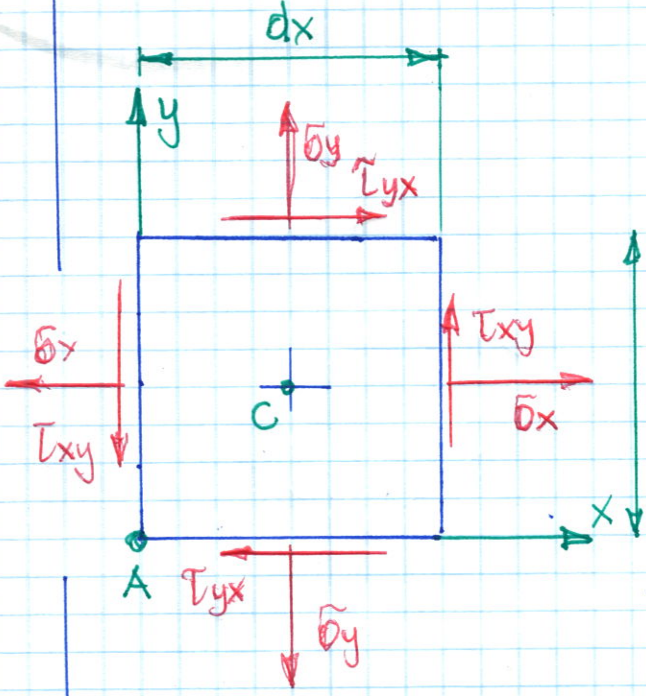
Principles of stress indexing
Zasady „indeksowania" naprężeń

⇓



- If the body is in equilibrium, then the cuboid is in equilibrium (after applying stresses)
- Jeśli ciało jest w równowadze to i prostopadłościan jest w równowadze (po przyłożeniu naprężeń)

Boltzmann axiom
Postulat Boltzmanna



$$\Sigma M_i^C = -\tau_{xy}\cdot dy\cdot dz\cdot\frac{dx}{2} + \tau_{yx}\cdot dx\cdot dz\cdot\frac{dy}{2} - \tau_{xy}\cdot dy\cdot dz\cdot\frac{dx}{2} + \tau_{yx}\cdot dx\cdot dz\cdot\frac{dy}{2} = 0 \quad \Big| : \frac{dx\,dy\,dz}{2}$$

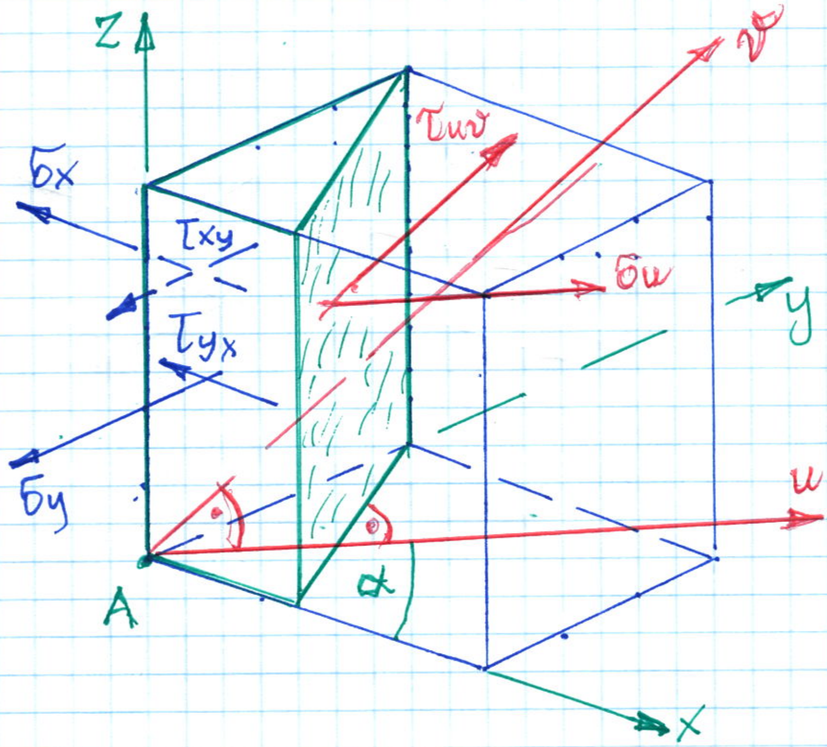
$$\Downarrow$$

$$-\tau_{xy}+\tau_{yx}-\tau_{xy}+\tau_{yx}=0$$

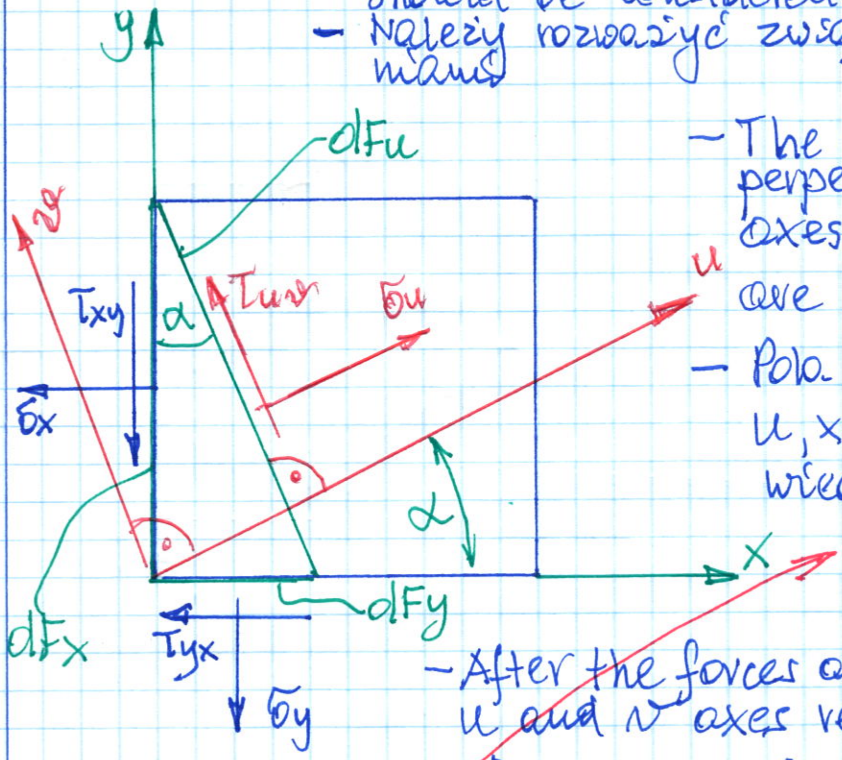
$$-2\tau_{xy}+2\tau_{yx}=0$$

$$\boxed{\tau_{xy}=\tau_{yx}}$$

- Stresses in diagonal section
  Rotational transformation
- Naprężenia w przekroju ukośnym. Transformacja obrotowa



- Now the relationships between stresses should be considered
- Należy rozważyć związki między naprężeniami



- The cross-sectional areas perpendicular to the axes u, x, y, respectively, are marked as: dFu, dFx, dFy
- Pola przekroju ⊥ do osi u, x, y oznaczono odpowiednio jako: dFu, dFx, dFy

- After the forces are projected on the u and v axes respectively
- Po zrzutowaniu sił na osie u oraz v odpowiednio:

$$\boxed{dF_x = dF_u \cdot \cos\alpha}\ ,\quad \boxed{dF_y = dF_u \cdot \sin\alpha}$$

① $\sum P_{iu} = \sigma_u \cdot dF_u - \sigma_x \cdot dF_x \cdot \cos\alpha - \tau_{xy}\, dF_x \cdot \sin\alpha$
$- \tau_{yx}\, dF_y \cdot \cos\alpha - \sigma_y\, dF_y \sin\alpha = 0$

② $\sum P_{iv} = \tau_{uv} \cdot dF_u + \sigma_x\, dF_x \sin\alpha - \tau_{xy}\, dF_x \cos\alpha +$
$\tau_{yx}\, dF_y \cdot \sin\alpha - \sigma_y \cdot dF_y \cos\alpha = 0$

① $\sum P_{iu} = \sigma_u \cdot dF_u - \sigma_x\, dF_u \cos^2\alpha - \tau_{xy}\, dF_u \cdot \cos\alpha \cdot \sin\alpha$
$- \tau_{yx}\, dF_u \sin\alpha \cdot \cos\alpha - \sigma_y \cdot dF_u \sin^2\alpha = 0 \quad \Big|\ : dF_u$

② $\sum P_{iv} = \tau_{uv} \cdot dF_u + \sigma_x \cdot dF_u \cos\alpha \cdot \sin\alpha - \tau_{xy}\, dF_u \cos^2\alpha$
$+ \tau_{yx}\, dF_u \sin^2\alpha - \sigma_y \cdot dF_u \sin\alpha \cdot \cos\alpha = 0 \quad \Big|\ : dF_u$

from ①	$\sigma_u = \sigma_x \cdot \cos^2\alpha + \sigma_y \cdot \sin^2\alpha + 2\tau_{xy} \cdot \sin\alpha \cdot \cos\alpha$
from ②	$\tau_{uv} = -(\sigma_x - \sigma_y) \sin\alpha \cdot \cos\alpha + \tau_{xy} (\cos^2\alpha - \sin^2\alpha)$

but / ale

$$\cos^2\alpha = \frac{1+\cos 2\alpha}{2}, \quad \sin^2\alpha = \frac{1-\cos 2\alpha}{2}$$

$$\sin\alpha \cdot \cos\alpha = \frac{\sin 2\alpha}{2}$$

$\Downarrow$

$$\boxed{\begin{aligned}\sigma_u &= \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos 2\alpha + \tau_{xy}\cdot\sin 2\alpha \\ \tau_{uv} &= -\frac{\sigma_x-\sigma_y}{2}\cdot\sin 2\alpha + \tau_{xy}\cdot\cos 2\alpha\end{aligned}}$$

for / gdy σ_v, τ_{vu}

for angle $= \alpha + 90^\circ$

$$\boxed{\begin{aligned}\sigma_v &= \frac{\sigma_x+\sigma_y}{2} - \frac{\sigma_x-\sigma_y}{2}\cos 2\alpha - \tau_{xy}\sin 2\alpha \\ \tau_{vu} &= \frac{\sigma_x-\sigma_y}{2}\sin 2\alpha - \tau_{xy}\cos 2\alpha\end{aligned}}$$

$$\tau_{uv} = -\tau_{vu}$$

- find the extreme of the function for σ_u, τ_{uv} respectively
- wyznaczyć ekstremum funkcji odpowiednio dla σ_u, τ_{uv}

a) $$\frac{d\sigma_u}{d\alpha} = 0 \Rightarrow \alpha = \alpha_0$$ principal normal stress

$$-(\sigma_x - \sigma_y)\cdot\sin 2\alpha + 2\tau_{xy}\cos 2\alpha = 0$$

$$\boxed{\operatorname{tg} 2\alpha_0 = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}} \Rightarrow \begin{aligned}\sigma_x &= \sigma_1 \\ \sigma_y &= \sigma_2 \\ \tau_{xy} &= 0 = \tau_{12}\end{aligned}$$

then / wówczas $\tau_{uv} = 0$!

When the extreme of the function σ_u, then $\tau_{uv} = 0$!

b) $$\boxed{\frac{d\tau_{uv}}{d\alpha} = 0} \Rightarrow \alpha = \alpha_1$$

$$-(\sigma_x - \sigma_y)\cos 2\alpha - 2\tau_{xy}\sin 2\alpha = 0$$

$$\boxed{tg\, 2\alpha_1 = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}}$$

$$tg\, 2\alpha_1 = -ctg\, 2\alpha_0$$

principal shear stress

$$\boxed{\alpha_1 = \alpha_0 + \frac{\pi}{4}}$$

Rotational transformation (for principal stresses)
Transformacja obrotowa (dla naprężeń głównych)

$$\boxed{\begin{aligned} \sigma_x \Leftarrow\ \sigma_u &= \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2}\cdot\cos 2\varphi \\ \sigma_y \Leftarrow\ \sigma_v &= \frac{\sigma_1 + \sigma_2}{2} - \frac{\sigma_1 - \sigma_2}{2}\cos 2\varphi \\ \tau_{xy} \Leftarrow\ \tau_{uv} &= -\frac{\sigma_1 - \sigma_2}{2}\cdot\sin 2\varphi \end{aligned}}$$

$\sigma_1, \sigma_2, \tau_{12} = 0$

$\sigma_u, \sigma_v, \tau_{uv}$ – ?

$\Downarrow$

$\sigma_x, \sigma_y, \tau_{xy}$ – ?

Mohr's circle (Otto Mohr)
(Koło Mohra)

$$\left(\sigma_u - \frac{\sigma_1 + \sigma_2}{2}\right)^2 = \left(\frac{\sigma_1 - \sigma_2}{2}\right)^2 \cos^2 2\varphi$$

$$+ \quad\quad\quad\quad +$$

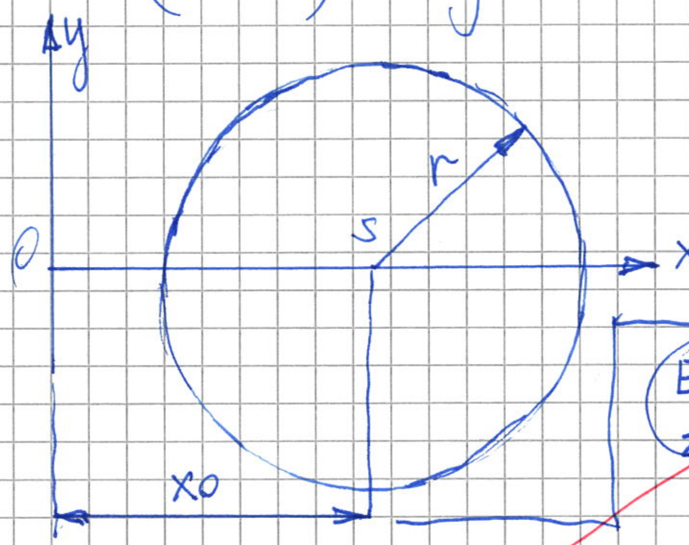
$$\tau_{uv}^2 = \left(\frac{\sigma_1 - \sigma_2}{2}\right)^2 \sin^2 2\varphi$$

$$\Downarrow$$

$$\boxed{\left(\sigma_u - \frac{\sigma_1 + \sigma_2}{2}\right)^2 + \tau_{uv}^2 = \left(\frac{\sigma_1 - \sigma_2}{2}\right)^2}$$

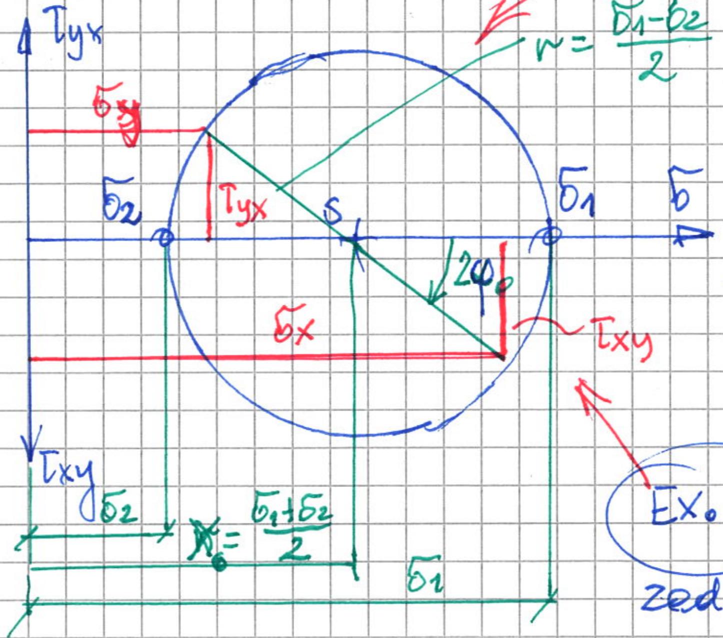
$$\left(\sigma_x - \frac{\sigma_1 + \sigma_2}{2}\right)^2 + \tau_{xy}^2 = \left(\frac{\sigma_1 - \sigma_2}{2}\right)^2$$

$$(x - x_0)^2 + y^2 = r^2$$



Ex. 1. zad. 1

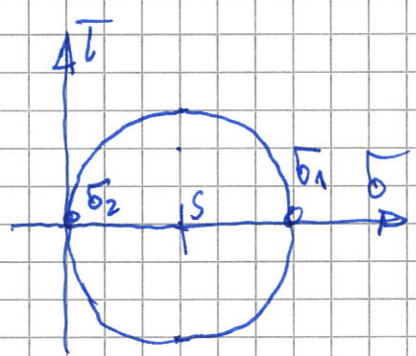
$$\frac{\varphi,\ \sigma_1,\ \sigma_2 \quad (\sigma_1 > \sigma_2)}{\sigma_x,\ \sigma_y,\ \tau_{xy} - ?}$$



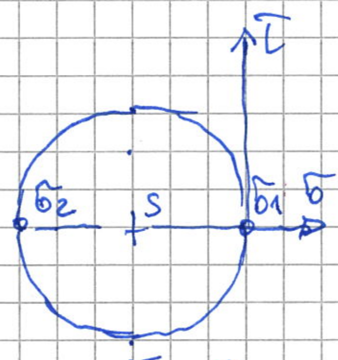
$$\left(\sigma_x - \frac{\sigma_1 + \sigma_2}{2}\right)^2 + \tau_{xy}^2 = \left(\frac{\sigma_1 - \sigma_2}{2}\right)^2$$

Ex. 2. zad. 2

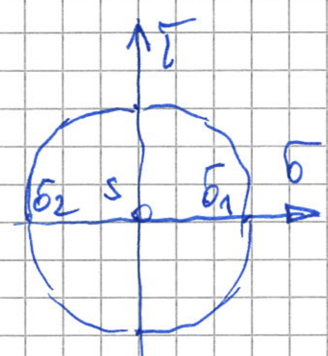
$$\frac{\sigma_x,\ \sigma_y,\ \tau_{xy}}{\sigma_1,\ \sigma_2,\ \alpha_0}$$



$\sigma_1 > 0$
$\sigma_2 = 0$
(uniaxial tension)
jednoosiowe rozciąganie



$\sigma_1 = 0$
$\sigma_2 < 0$
(uniaxial compression)
jednoosiowe ściskanie



$\sigma_1 > 0$
$\sigma_2 < 0$
$|\sigma_1| = |\sigma_2|$

2.

Homework / zadanie domowe

Ex. 1. / zad. 1.

$\sigma_1 = 600$ MPa, $\sigma_2 = -400$ MPa

$\varphi = 15°$

$\sigma_x, \sigma_y, \tau_{xy}$ - ?

(analytically and graphically)

Ex. 2 / zad. 2.

$\sigma_x = 400$ MPa, $\sigma_y = -200$ MPa

$\tau_{xy} = 300$ MPa

$\sigma_1, \sigma_2, \alpha_0$ - ?

(analytically and graphically)