Column buckling

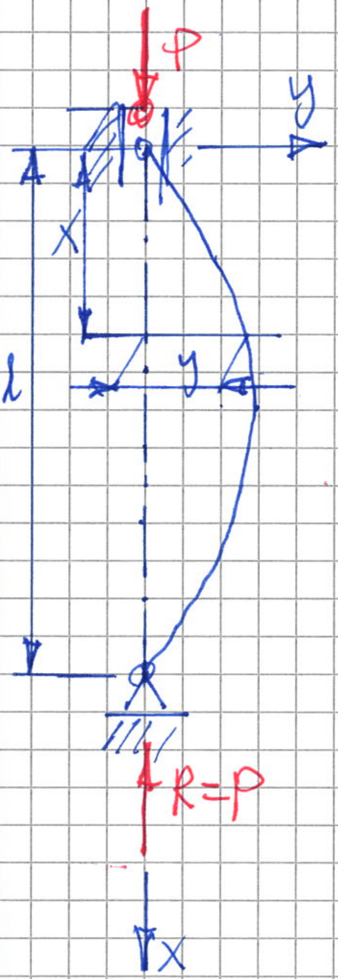
Wyboczenie kolumny / pręta
słup

Column buckling is perhaps the only area of structural mechanics in which failure is not related to the strength of the material. A column buckling analysis consists of determining the maximum load a column can support before it collapses. For long columns the collapse has nothing to do with material yield. It is instead governed by the column's stiffness.

Wyboczenie pręta jest prawdopodobnie jedynym obszarem mechaniki, w którym zniszczenie nie jest związane z wytrzymałością materiału. Analiza wyboczenia pręta polega na określeniu maksymalnego obciążenia, jakie może wytrzymać słup, zanim przybierze nową postać. W przypadku długich prętów wyboczenie nie ma nic wspólnego z wytrzymałością materiału. Istotny wpływ ma natomiast sztywność pręta.

Euler buckling theory

Wyboczenie sprężyste pręta (Eulera)



Beam bending theory is central to column buckling analyses using Euler's theory

- Internal bending moment in a loaded and deformed column is

$$M = P \cdot y$$

- Beam bending equation is (Równanie różniczkowe osi ugiętej)

$$EI \frac{d^2y}{dx^2} = -M = -P \cdot y$$

- this produces the following equation

$$EI \frac{d^2y}{dx^2} + Py = 0 \quad \Big| : EI$$

$$\frac{d^2y}{dx^2} + \frac{P}{EI} \cdot y = 0 \qquad \text{but} \quad \frac{P}{EI} = k^2$$

$$\boxed{\frac{d^2y}{dx^2} + k^2 \cdot y = 0}$$

general integral of this equation

$$y = A \cdot \sin k \cdot x + B \cdot \cos k \cdot x$$

Boundary conditions:

(1) $y(x=0) = 0 \Rightarrow A \cdot \sin k \cdot 0 + B \cdot \overbrace{\cos k \cdot 0}^{=1} \Rightarrow B = 0$

(2) $y(x=l) = 0$

$$\boxed{y = A \cdot \sin k \cdot x}$$

(2) $\Rightarrow$ $\boxed{A \cdot \sin k \cdot l = 0}$

This condition is met in two cases:

- when $A = 0$ (for each value of x, $y = 0$) so-called a trivial case) (przypadek trywialny)
- when $\sin kl = 0 \Rightarrow kl = n \cdot \pi$, $n = 0, 1, 2, 3, \ldots, n$

$$\Downarrow$$

$$kl = \sqrt{\frac{P}{EI}} \cdot l = n\pi$$

$$\Downarrow$$

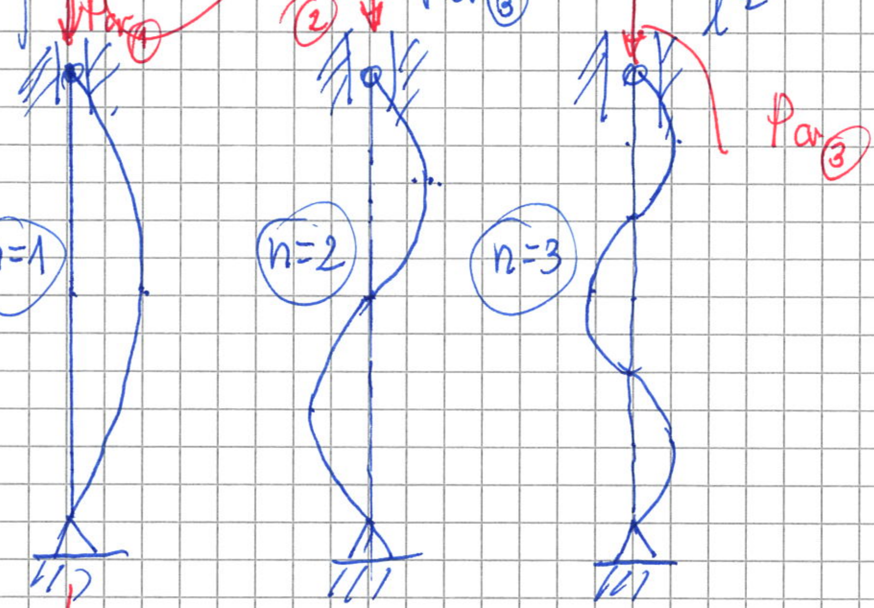
$$\boxed{P = \frac{n^2 \pi^2 \cdot EI}{l^2}}$$

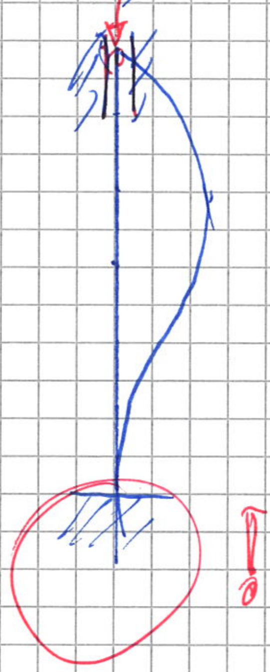
- for $n = 0 \Rightarrow$ trivial case
- for $n = 1 \Leftrightarrow$ $\boxed{P_{critical} = P_{cr_{(1)}} = \frac{\pi^2 \cdot EI}{l^2}}$

$n = 1$ $\Uparrow$

for $n = 2 \Longrightarrow P_{cr(2)} = \frac{4\pi^2 EI}{l^2}$

for $n = 3 \Longrightarrow P_{cr(3)} = \frac{9\pi^2 EI}{l^2}$





$$P_{cr(1)} = \frac{\pi^2 \cdot EI}{l_r^2} \qquad l_r = \alpha \cdot l$$

α – depends on type of support

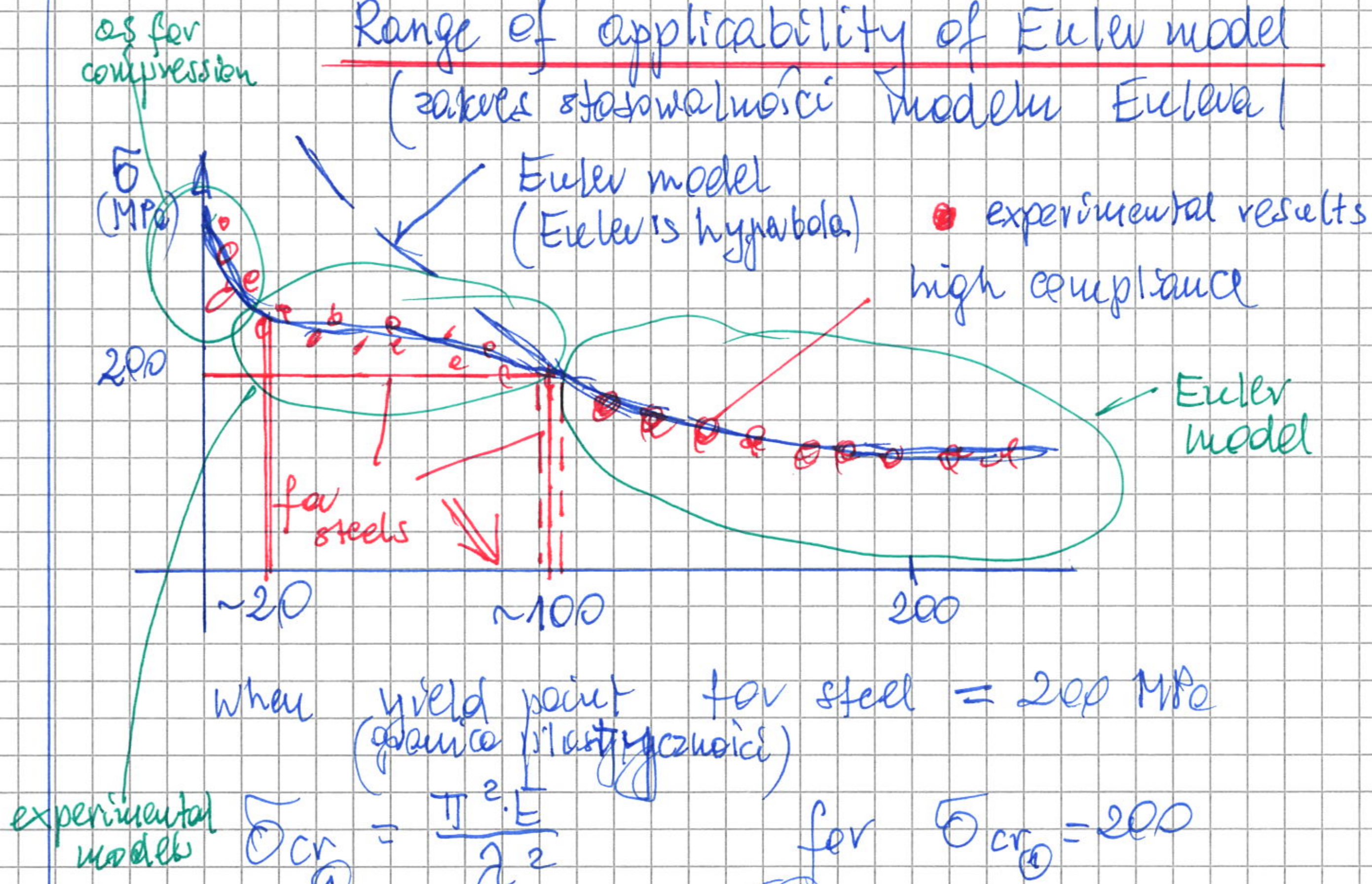
$$\sigma_{cr(1)} = \frac{\pi^2 \cdot EI}{l^2 \cdot A} \quad \text{but} \quad \frac{I}{A} = i^2$$

i – radius of gyration (promień bezwładności)

$\frac{l}{i} = \lambda$ slenderness ratio (smukłość)

$$\boxed{\sigma_{cr(1)} = \frac{\pi^2 \cdot E}{\lambda^2}} \qquad \sigma_{cr(n)} = \frac{n^2 \cdot \pi^2 \cdot E}{\lambda^2}$$

Range of applicability of Euler model
(zakres stosowalności modelu Eulera)



when yield point for steel = 200 MPa
(granica plastyczności)

$$\sigma_{cr_{1}} = \frac{\pi^2 \cdot E}{\lambda^2} \quad \text{for } \sigma_{cr_{1}} = 200$$

$$\lambda^2 = \frac{\pi^2 \cdot E}{\sigma_{cr_{1}}} = \frac{\overset{\pi^2}{\sim 10} \cdot 2 \cdot 10^5}{200} \cong 10000$$

$\Downarrow$

$\lambda_l \cong 100$ — λ_l – limit value of slenderness

range of applicability
for $\lambda \geq \lambda_l$ (100 for steel) $\Rightarrow$ Euler model

for $\lambda \leq \lambda_l$ — experimental model (engineering)
($20 \leq \lambda \leq 100$ for steel)

I Tetmajer – Jasiński model
$\sigma_{cr_{1}} = a - b\lambda$ — a, b – from experiment

II Johnson – Ostenfeld model
$\sigma_{cr_{1}} = A - B\lambda^2$ — A, B – from experiment

Cross-section reduction factor method
"uniwersal metod"
independent of λ

Metoda współczynnika zmniejszającego przekrój
metoda "uniwersalna", niezależna od λ

k_c – strength limit for compression
(dopuszczalne naprężenie na ściskanie)

$$\sigma_{cr} \leq k_c \cdot \beta \quad (1)$$

β – reduction factor of cross-section

$$\Downarrow$$

or $$\frac{P_{cr}}{A \cdot \beta} \leq k_c$$

β depends on λ and type of material

slenderness (smukłość) λ	steel (yield point) Re	
	Re ≤ 260 MPa	300 ≤ Re ≤ 380
0	1	1
50	0,85	0.83
100	0,55	0.44
150	0.28	0.20
200	0,16	0.11

ıd ıs not only resısted by arch rıb but a percentage of the load ıs also resısted by botto

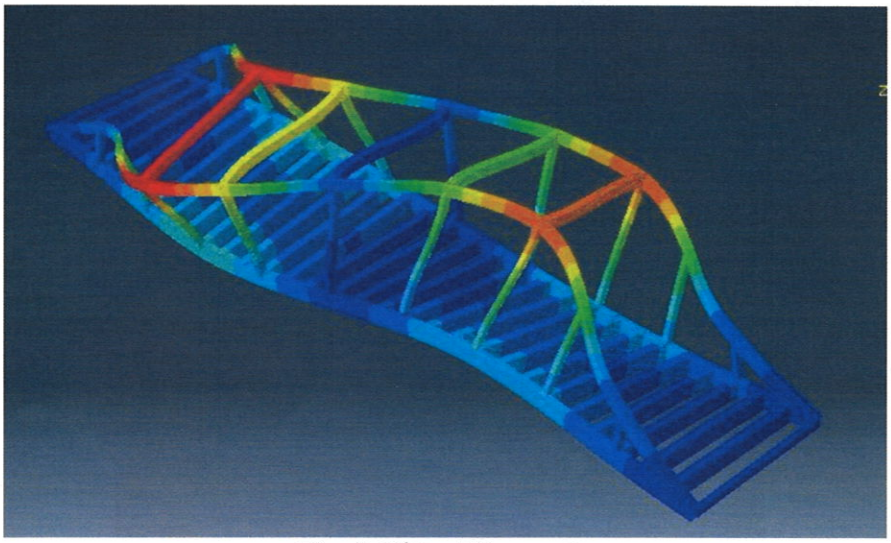
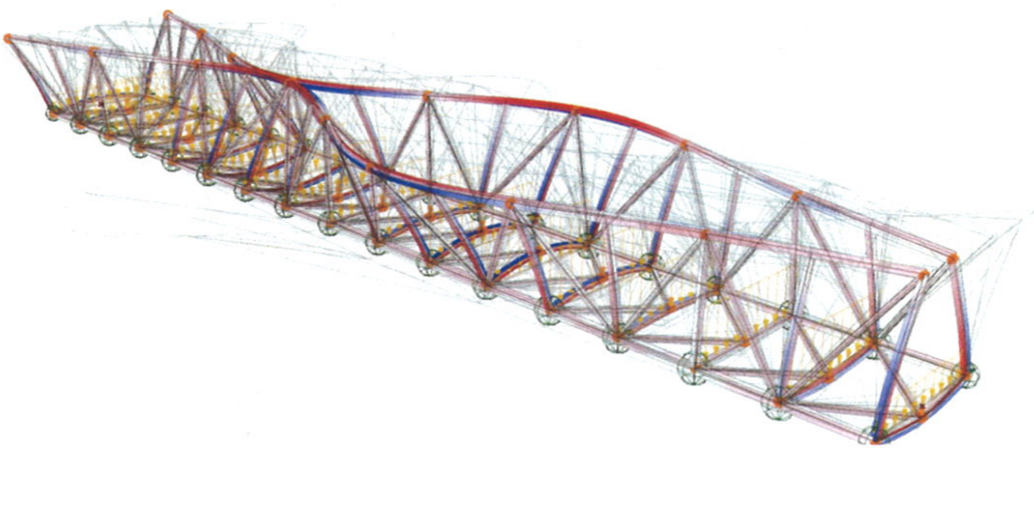
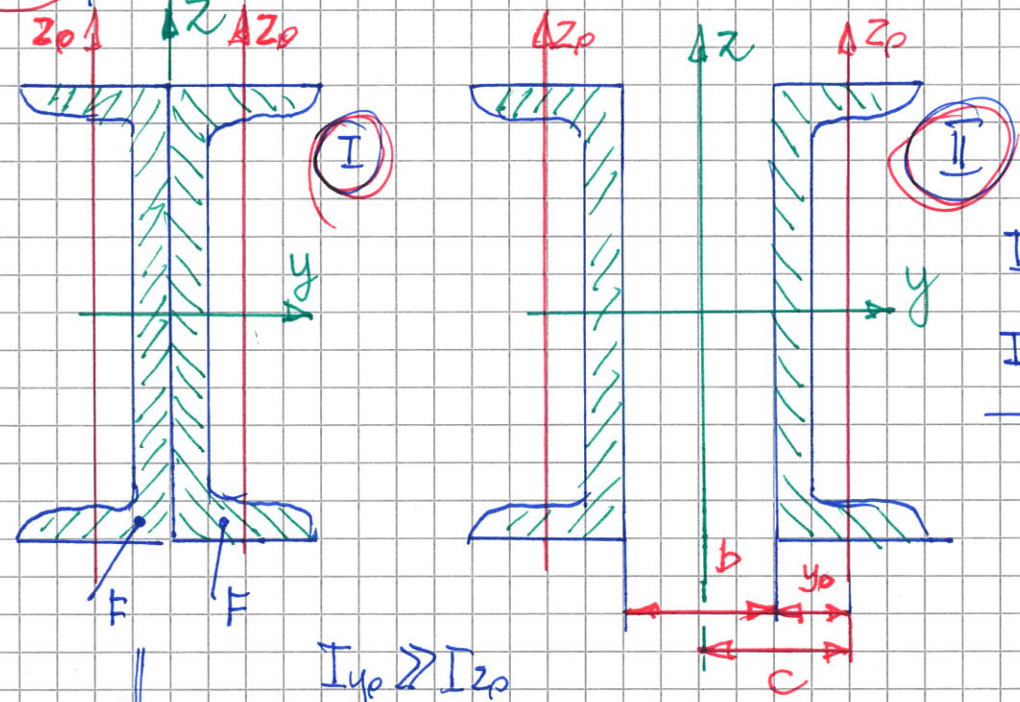


Fig. 4: 1st buckling mode



Ex. 1. / ćw. 1



for one [element

I_{y0}, W_y, i_y

I_{z0}, W_{z0}, i_{y0}

b, c - ?

$I_{y0} \gg I_{z0}$

for][

$I_{min} = I_z = 2\left(I_{z0} + F\cdot y_0^2\right)$

$I_{max} = I_y = 2\, I_{y0}$

for which value of b moment of inertia I_z will be as large as I_y $\Rightarrow$

$2\left(I_{z0} + F\cdot c^2\right) = 2 I_{y0}$

but $c = \frac{b}{2} + y_0$

$c^2 = \left(\frac{b}{2} + y_0\right)^2$

$\cancel{2}\left[I_{z0} + F\left(\frac{b}{2} + y_0\right)^2\right] = \cancel{2} I_{y0}$

$I_{z0} + F\left(\frac{b}{2} + y_0\right)^2 = I_{y0}$

$\left(\frac{b}{2} + y_0\right)^2 = \frac{I_{y0} - I_{z0}}{F}$, $\quad \frac{b}{2} + y_0 = \sqrt{\frac{I_{y0} - I_{z0}}{F}}$

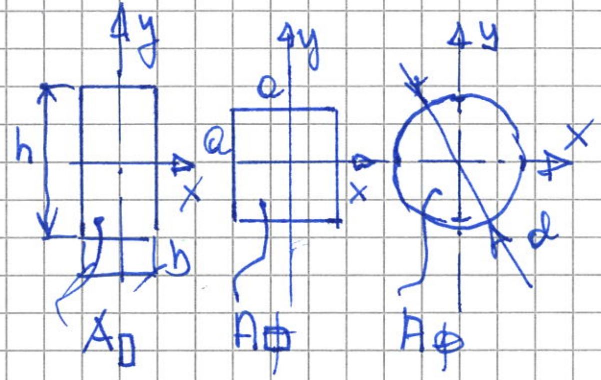
$b = \left(\sqrt{\frac{I_{y0} - I_{z0}}{F}} - y_0\right)\cdot 2$

for] [, b

$I_z = I_y$

$$\boxed{b = 2\left(\sqrt{\frac{I_{y0} - I_{z0}}{F}} - y_0\right)}$$

Ex. 2. / ćw. 2



$A_{\square} = A_{\boxplus} = A_{\phi} = A$

$h = 2b$, $l_{\square} = l_{\boxplus} = l_{\phi} = l$

E, Euler's theory

$\lambda \geq \lambda_{cr}$

$\sigma_{cr\square} : \sigma_{cr\boxplus} : \sigma_{cr\phi} = ?$

$$\sigma_{cr(1)} = \frac{\pi^2 \cdot E I}{l^2 \cdot A} \; ; \quad \underbrace{\frac{I}{A} = i^2, \; \frac{l}{i} = \lambda}_{\Downarrow}$$

$$\sigma_{cr(1)} = \frac{\pi^2 \cdot E}{\lambda^2}$$

for $\square$ (h, b)

$$I_x = \frac{bh^3}{12}$$

$$I_y = \frac{hb^3}{12}$$

$$I_y < I_x$$

$$I_y = I_{min}$$

$$h = 2b$$

$$A = h \cdot b = 2b \cdot b = 2b^2$$

$$b = \sqrt{\frac{A}{2}}$$

$$h = 2\sqrt{\frac{A}{2}}$$

for $\boxplus$ (a)

$$I_x = I_y = \frac{a^4}{12}$$

$$A = a^2 \Rightarrow a = \sqrt{A}$$

$$I_x = I_y = \frac{(\sqrt{A})^4}{12} = \frac{A^2}{12} = I_{\boxplus}$$

for ϕ

$$I_x = I_y = \frac{\pi d^4}{64}$$

$$A = \frac{\pi d^2}{4}$$

$$d = \sqrt{\frac{4A}{\pi}}$$

$$I_x = I_y = \frac{\pi}{64}\left(\sqrt{\frac{4A}{\pi}}\right)^4 = \frac{\pi}{64} \cdot \frac{16A^2}{\pi^2} = \frac{A^2}{4\pi} = I_{\phi}$$

$$I_y = \frac{h b^3}{12} = \frac{1}{12} 2\sqrt{\frac{A}{2}} \sqrt{\frac{A^3}{8}} = \frac{1}{6}\sqrt{\frac{A^4}{16}} = \frac{A^2}{24} = I_{y\square\,min}$$

-8

$$\sigma_{cr\,\square} : \sigma_{cr\,\square} : \sigma_{cr\,\oplus} = I_{min\,\square} : I_{\square} : I_{\oplus}$$

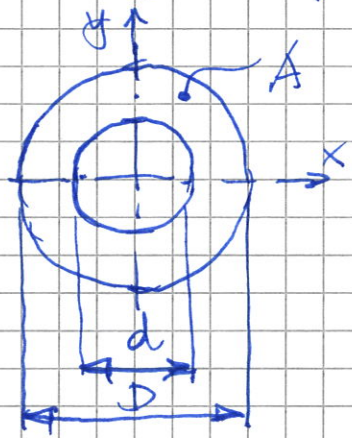
$$= \frac{A^2}{24} : \frac{A^2}{12} : \frac{A^2}{4\pi} = \frac{1}{24} : \frac{1}{12} : \frac{1}{4\pi} =$$

$$= \frac{1}{24} : \frac{1}{12} : \frac{1}{12{,}56}$$

1 3 2

order of buckling
(kolejność wyboczenia)

additional:



$$D = 2d$$

$$I_x = I_y = \frac{\pi (D^4 - d^4)}{64}$$

$$A = \frac{\pi (D^2 - d^2)}{4}$$

$$A = \frac{\pi}{4}(4d^2 - d^2) = \frac{\pi \cdot 3d^2}{4}$$

$$d = \sqrt{\frac{4A}{3\pi}} \quad ; \quad D = 2\sqrt{\frac{4A}{3\pi}}$$

$$I_{\oplus} = \frac{\pi}{64}\left[2^4\left(\sqrt{\frac{4A}{3\pi}}\right)^4 - \left(\sqrt{\frac{4A}{3\pi}}\right)^4\right] =$$

$$= \frac{\pi}{64}\left[16\left(\frac{4A}{3\pi}\right)^2 - \left(\frac{4A}{3\pi}\right)^2\right] = \frac{\pi}{64} \cdot 15 \cdot \frac{16A^2}{9\pi^2} =$$

$$= \frac{15A^2}{36\pi} \quad \Big| : A^2 \cdot 15 \quad \longrightarrow \quad \frac{1}{7{,}539} \qquad 4$$

$$= \cancel{\frac{1}{23{,}68}} \qquad \cancel{2^*}$$

$\frac{I}{A} = i^2$ $\quad$ $i = \sqrt{\frac{I}{A}}$ $\quad$ l_{min} for $\lambda_{cr} = 100$

$$i_{\square} = \sqrt{\frac{A^2}{24A}} = \sqrt{\frac{A}{24}} = 0{,}204\sqrt{A}$$

$$i_{\boxplus} = \sqrt{\frac{A^2}{12A}} = \sqrt{\frac{A}{12}} = 0{,}288\sqrt{A}$$

$$i_{\phi} = \sqrt{\frac{A^2}{4\pi A}} = \sqrt{\frac{A}{4\pi}} = 0{,}282\sqrt{A}$$

$$i_{\oplus} = \sqrt{\frac{15A^2}{36\pi A}} = \sqrt{\frac{15A}{36\pi}} = 0{,}364$$

$\lambda_{cr} = 100$ $\quad$ $\frac{l}{i} = \lambda_{cr} \Rightarrow l = i \cdot \lambda_{cr}$

$l_{\square} = 204\sqrt{A}$, $l_{\boxplus} = 288\sqrt{A}$, $l_{\phi} = 282\sqrt{A}$

$l_{\oplus} = 364\sqrt{A}$