Shear stress by uneven bending
Naprężenie tnące przy zginaniu nierównomiernym
(Horizontal shear stress by bending)

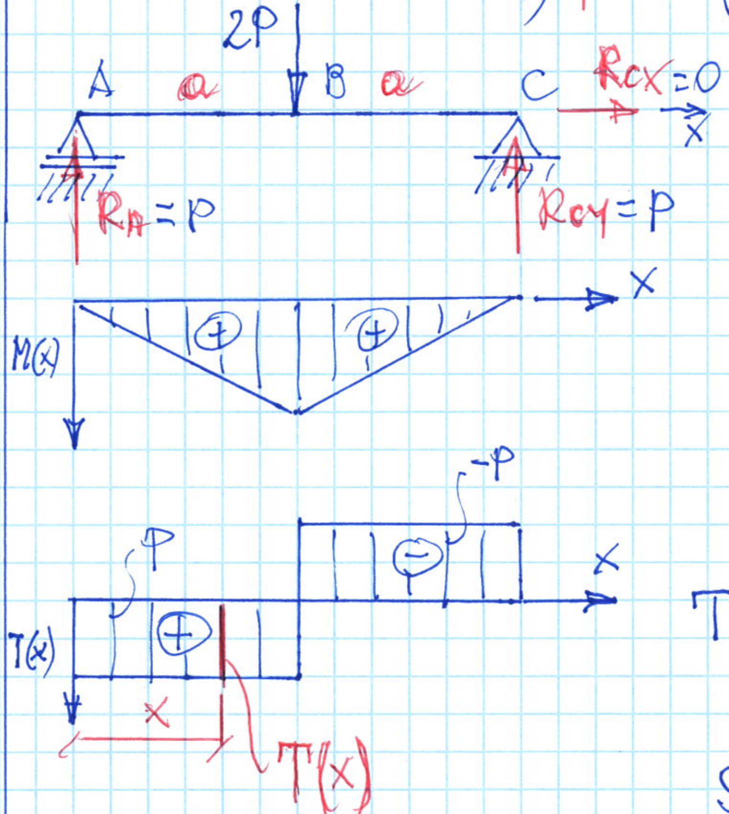
Dmitrii Ivanovich Zhuravskii
1821-1891
- Russian pioneer of bridge construction

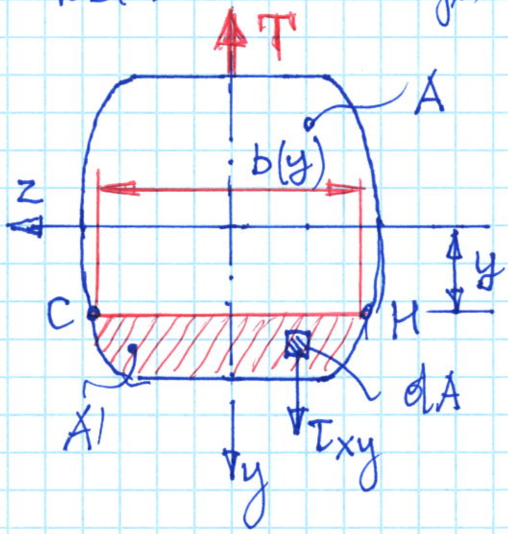
The Zhuravskii Shear Stress formula
(wzór Żurawskiego na naprężenie tnące w zginaniu nierównomiernym)

If $M(x) \neq const$ then $T(x) = \frac{dM(x)}{dx}$

$$\boxed{\tau_{xy} = \frac{T(x) \cdot S_z}{I_z \cdot b(y)}}$$

Zhuravskii formula (1855)
(Jourawski's formula
wzór Żurawskiego)





$$S_z = \int_{A'} y \, dA'$$

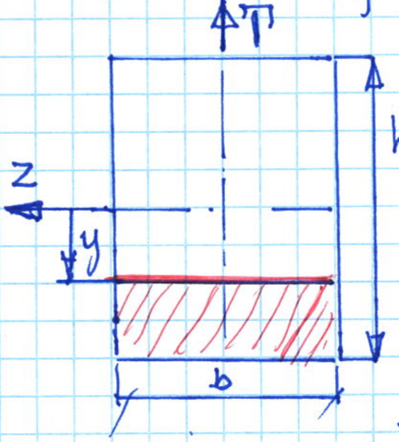
$T(x)$ – total shear force at the location in question (siła tnąca w analizowanym przekroju)

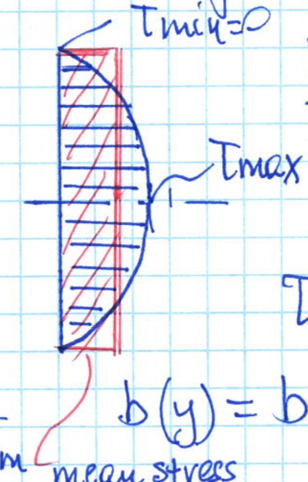
S_z – statical moment of area A' (moment statyczny odciętej części A')

I_z – moment of inertia of the entire cross sectional area (moment bezwładności całego przekroju)

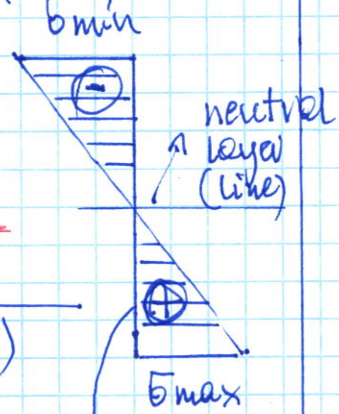
$b(y)$ – thickness in the material perpendicular to the shear (szerokość przekroju w odległości y)

Ex. 1. Shear stress distribution along a horizontal line of a rectangular cross section





mean stress (naprężenie średnie)



distribution of normal stress

$$\frac{b,\ h,\ T}{\tau_{xy}(y) - ?}$$

$$\tau_{xy} = \frac{T \cdot S_z}{I_z \cdot b(y)}$$

$$b(y) = b$$

$$I_z = \frac{b h^3}{12}$$

$$S_z = b \cdot \frac{h}{2} \cdot \frac{h}{4} - b \cdot y \cdot \frac{y}{2} = \frac{bh^2}{8} - \frac{by^2}{2} = \frac{b}{2}\left(\frac{h^2}{4} - y^2\right)$$

$$\tau_{xy} = \frac{T \cdot \frac{\cancel{b}}{2}\left(\frac{h^2}{4} - y^2\right)}{\frac{\cancel{b}h^3}{12} \cdot b} = \frac{12T}{2h^3 \cdot b}\left(\frac{h^2}{4} - y^2\right) = \frac{6T}{bh^3}\left(\frac{h^2}{4} - y^2\right)$$

$$\boxed{\tau_{xy} = \frac{6T}{bh^3}\left(\frac{h^2}{4} - y^2\right)}$$

for $y = 0$ $\quad \tau_{xy} = \tau_{max} = \frac{6T}{bh^3} \cdot \frac{h^2}{4} = \frac{3}{2}\frac{T}{b \cdot h}$

for $y = \frac{h}{2}$ $\quad \tau_{xy} = \tau_{min} = 0$

τ_m – mean stress $= \dfrac{T}{b \cdot h}$

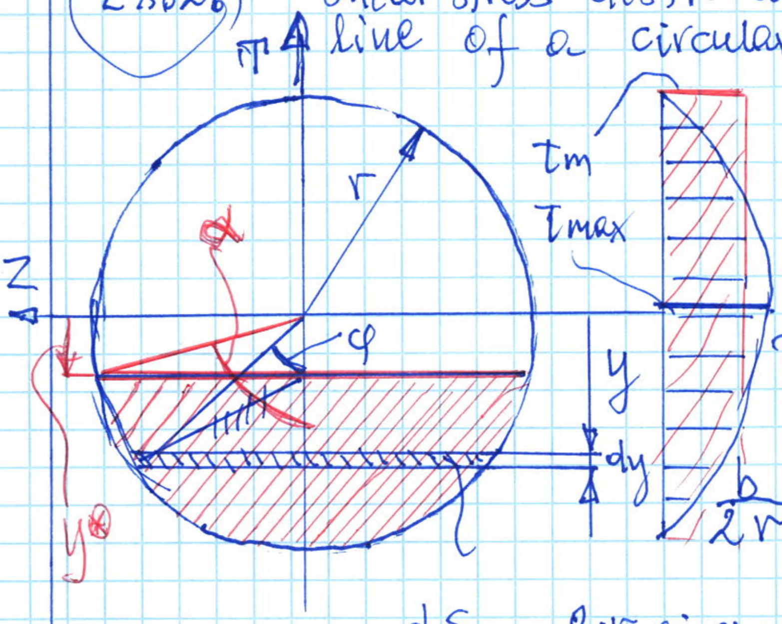
$$\tau_{max} = \frac{3}{2}\tau_m$$

for square section $b = h = a$

$$\tau_{xy} = \frac{6T}{ah^3}\left(\frac{a^2}{4} - y^2\right)$$

$$\tau_{\substack{max \\ (y=0)}} = \frac{3}{2}\frac{T}{a^2}$$

Ex. 2. Shear stress distribution along a horizontal line of a circular cross section.



$$\tau_{xy} = \frac{T \cdot S_z}{I_z \cdot b(y^*)}$$

$$\frac{b}{2r} = \sin\alpha \qquad b = 2 \cdot r \cdot \sin\alpha$$

$$dS_z = 2 \cdot r \cdot \sin\varphi \cdot y \cdot dy = 2 \cdot r \cdot y \cdot \sin\varphi \cdot dy$$

$$y = r \cdot \cos\varphi \rightarrow dy = -r \sin\varphi \, d\varphi$$

$$S_z = -\int_{\alpha}^{0} 2ry \sin\varphi \cdot r\sin\varphi \, d\varphi = -\int_{\alpha}^{0} 2 r \cdot r\cos\varphi \sin\varphi \cdot r \cdot \sin\varphi \, d\varphi =$$

$$= -2r^3 \int_{\alpha}^{0} \sin^2\varphi \cos\varphi \, d\varphi = -2r^3 \int_{\alpha}^{0} \sin^2\varphi \, d(\sin\varphi) =$$

$\sin\varphi = x$

$$= -\frac{2}{3} r^3 \sin^3(\varphi)\Big|_{\alpha}^{0} = \frac{2}{3} r^3 \sin^3\alpha$$

$$\boxed{S_z = \frac{2}{3} r^3 \sin^3\alpha}$$

$$I_o = \frac{\pi r^4}{2} \Rightarrow I_z = \frac{I_o}{2} = \frac{\pi r^4}{4}$$

$$\tau_{xy} = \frac{T \cdot \cancel{\frac{2}{3}} r^3 \sin^3\alpha}{\frac{\pi r^4}{4} \cdot \cancel{2} \cdot r \cdot \sin\alpha} = \frac{4T \cdot r^3 \cdot \sin^3\alpha}{\pi r^4 \cdot r \cdot \sin\alpha}$$

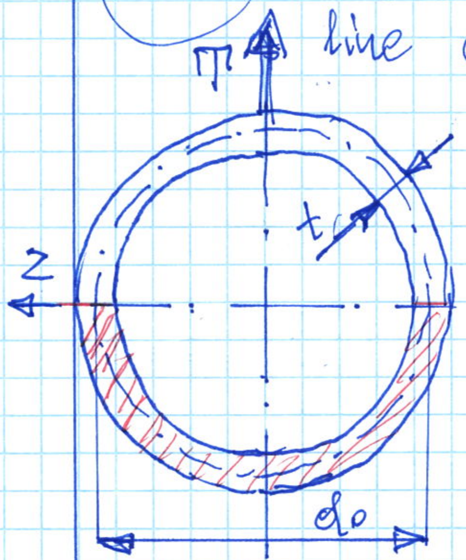
$$\boxed{\tau_{xy} = \frac{4}{3} \frac{T}{\pi r^2} \sin^2\alpha}$$

for $\alpha = 0$ $\quad \tau_{xy} = \tau_{min} = 0$

for $\alpha = \frac{\pi}{2}$ $\quad \tau_{xy} = \tau_{max} = \frac{4}{3}\frac{T}{\pi r^2}$

$\tau_m = \frac{T}{\pi r^2}$ (mean) $\qquad \tau_{max} = \frac{4}{3}\tau_m$

Ex. 3. Shear stress ~~distribution~~ maximum along a horizontal line of a ~~circular~~ tubular cross section.



$$\frac{T,\ d_0,\ t}{\tau_{xy\,max} - ?} \qquad t \ll d_0$$

$$\tau_{max} = \frac{T \cdot S_{z(max)}}{I_z \cdot b(y)}$$

$$I_z = \frac{\pi}{4}\left[\left(r_0 + \frac{t}{2}\right)^4 - \left(r_0 - \frac{t}{2}\right)^4\right] =$$

$r_0 = \frac{d_0}{2}$

$$= \cancel{\qquad\qquad} = \pi r_0^3 \cdot t\left(1 + \frac{t^2}{4r_0^2}\right)$$

~~for $t \ll d_0$~~

for $t \ll r_0$

$$\cong \pi r_0^3 t \qquad \boxed{I_z \cong \pi r_0^3 t} = \frac{\pi d_0^3 t}{8}$$

$$\underset{\text{(for semicircle)}}{S_{z\,max}} = \frac{2}{3} r^3$$

$$\underset{\text{(for tube)}}{S_{z\,max}} = \frac{2}{3}\left[\left(r_0 + \frac{t}{2}\right)^3 - \left(r_0 - \frac{t}{2}\right)^3\right] = 2r_0^2 \cdot t\left[1 + \frac{t^2}{12 r_0^2}\right]$$

for $t \ll r_0$

$$\cong 2r_0^2 \cdot t = 2\,\frac{d_0^2}{4} \cdot t = \frac{d_0^2 \cdot t}{2}$$

$$\underset{(t \ll d_0)}{\tau_{max}} \cong \frac{T \cdot \frac{d_0^2 \cdot t}{2}}{\frac{\pi d_0^3 \cdot t}{8} \cdot 2t} = \frac{2T \cdot d_0^2 \cdot t}{\pi d_0^3 \cdot t^2} = \frac{2T}{\pi d_0 \cdot t}$$

$$\boxed{\tau_{max} \cong \frac{2T}{\pi d_0 \cdot t}}$$