Castigliano theorem – 2. Hyperstatic beams (Method of Least Work)

Since the deflection of the point of application of the hyperstatic generalised force $P_i = R_i$ is zero, by applying the Castigliano's theorem, we can write

$$\boxed{\frac{\partial U}{\partial R_i} = p_i = 0}$$

This equation states that the first partial derivative of the strain energy with respect to the R_i (redundant) must be equal to zero.

This implies that for the value of the redundant that satisfies the equation of equilibrium and campability, the strain energy of the structure is a minimum or maximut

but

Since for a linearly elastic, there is no maximum value of strain energy, because it can be increased indefinitely by increasing the value of redundant, we conclude that for the true value of the redundant the strain energy must be a minimum

If a structure is indeterminate / hyperstatic to the n^{th} degree, the n redundants / hyperstatic forces are selected, and the strain energy for the structure is expressed in terms of the known external loading and the n unknown redundants as

$$U = f(w, R_1, R_2, R_3, \ldots, R_n)$$

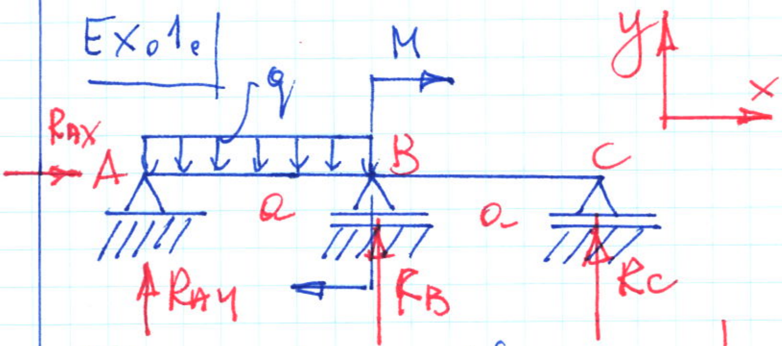
in which w represents all the known loads and $R_1, R_2, R_3, \ldots, R_n$ denote the n redundants

Next, the principle of least work is applied separately for each redundant (hyperstatic ~~generalized~~ generalised force) by partially differentiating the strain energy expressions with respect to each of the redundants and by setting each partial derivative equal to zero, that is,

$$\boxed{\frac{\partial U}{\partial R_1}=0,\ \frac{\partial U}{\partial R_2}=0,\ \frac{\partial U}{\partial R_3}=0,\ \dots,\ \frac{\partial U}{\partial R_n}=0}$$

which represents a system of n simultaneous equation in terms of n redundants and can be solved for the redundants

Ex. 1.



$q,\ a,\ M=qa^2,\ EI=\text{const}$

$R_{Ax},\ R_{Ay},\ R_B,\ R_C$ – ?

Menabrea–Castigliano's theorem

(I) static equations

(1) $\sum P_{ix} = R_{Ax} = 0$

(2) $\sum P_{iy} = R_{Ay} - qa + R_B + R_C = 0$

(3) $\sum M_i^C = R_{Ay}\cdot 2a - qa\cdot \frac{3}{2}a + M + R_B\cdot a = 0$

$$\boxed{(4)\ \frac{\partial U}{\partial R_{Ay}} = y_A = 0}$$

add. geometrical eq.

4 reactions – 3 static eqs. ⟹ 1× hyperstatic beam

We choose/select R_{Ay} as the hyperstatic force

Then we have to find a relationship/function

$$R_B = f(\text{active forces},\ R_{Ay})$$

from the static equation nr 3 ⟹

$$R_B = \frac{1}{a}\left(-R_{Ay}\,2a + \frac{3}{2}qa^2 - M\right) = -2R_{Ay} + \frac{3}{2}qa - \frac{M}{a}$$

and respectively

$0 \le x_1 \le a$

$$M_1(x) = R_{Ay} \cdot x - \frac{qx^2}{2} \quad ; \quad \frac{\partial M_1(x)}{\partial R_{Ay}} = x$$

$a \le x_2 \le 2a$

$R_B = f(\ldots, R_{Ay})$

$$M_2(x) = R_{Ay} \cdot x - qa\left(x - \frac{a}{2}\right) + M + R_B(x-a) =$$

$$R_{Ay} \cdot x - qa\left(x - \frac{a}{2}\right) + M + \left(-2R_{Ay} + \frac{3}{2}qa - \frac{M}{a}\right)(x-a)$$

$$\frac{\partial M_2(x)}{\partial R_{Ay}} = x - 2(x-a) = x - 2x + 2a = -x + 2a$$

(4) $$\frac{\partial U}{\partial R_{Ay}} = \frac{1}{EI}\left[\int_0^a M_1(x) \cdot \frac{\partial M_1(x)}{\partial R_{Ay}} dx + \int_a^{2a} M_2(x) \cdot \frac{\partial M_2(x)}{\partial R_{Ay}} dx = \right.$$

$$= y_A = 0 \;!$$

$$\Downarrow$$

$$\int_0^a \left(R_{Ay} \cdot x - \frac{qx^2}{2}\right) \cdot x \cdot dx + \int_a^{2a} \left[R_{Ay} \cdot x - qa\left(x - \frac{a}{2}\right) + \left(-2R_{Ay} + \frac{3}{2}qa - \frac{M}{a}\right)(x-a)\right](-x+2a)\, dx = 0 \;!$$

and after carrying out all the transformations and calculations we will get the value of hyperstatic reaction R_{Ay} because that's the only unknown in this equation.

Then we calculate the remaining reactions R_B, R_C from the static equations (1) and (2)

but keep attention!

(1) — we can't select reaction R_B as the hyperstatic force

then

$$④ \quad \frac{\partial U}{\partial R_B} = y_B = 0$$

then, when writing the equations for the bending moments (going from the left side), it is necessary to determine in advance

$$R_{AY} = f\,(q, M, R_B)$$

② – we can also select reaction R_C as the hyperstatic force

then

$$④ \quad \frac{\partial U}{\partial R_C} = y_C \neq 0$$

then, when writing the equations for the bending moments (going from the left side), it is necessary to determine in advance

$$R_{AY} = f^{*}(q, M, R_C)$$

$$R_B = f^{**}(q, M, R_C)$$

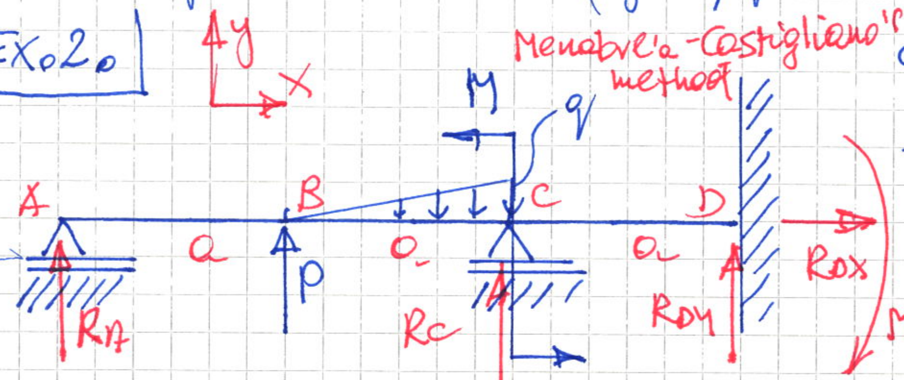
It can therefore be concluded that the choice/selection of R_{AY} reaction or R_B reaction as hyperstatic force is the optimal decision (going from the left side).

Ex. 2.

Menabrea–Castigliano's method

$q, a, P = qa, M = qa^2$
$EI = const$

$R_A, R_C, R_{DX}, R_{DY}, M_D$ – ?



5 reactions – 3 static equations ⇒ 2x hyperstatic beam

(I) static eqs.

(1) $\sum R_x^i = 0$

(2) $\sum R_y^i = R_A + P - \frac{1}{2}qa + R_C + R_{DY} = 0$

(3) $\sum M_i^D = R_A \cdot 3a + P \cdot a - \frac{1}{2}qa \cdot \frac{4}{3}a - M + R_C \cdot a + M_D \neq 0$

We select R_A and R_C as the hyperstatic forces

(4) $y_A = \frac{\partial U}{\partial R_A} = 0$

(5) $y_C = \frac{\partial U}{\partial R_C} = 0$

$0 \leq x_1 \leq a$

$M_1(x) = R_A \cdot x \quad ; \quad \frac{\partial M_1(x)}{\partial R_A} = x \quad ; \quad \frac{\partial M_1(x)}{\partial R_C} = 0$

$a \leq x_2 \leq 2a$

$M_2(x) = R_A \cdot x + P(x-a) - q\frac{(x-a)^3}{6a}$

$\frac{\partial M_2(x)}{\partial R_A} = x \quad ; \quad \frac{\partial M_2(x)}{\partial R_C} = 0$

$2a \leq x_3 \leq 3a$

$M_3(x) = R_A \cdot x + P(x-a) - \frac{1}{2}qa\left(x - \frac{5}{3}a\right) - M + R_C(x-2a)$

$\frac{\partial M_3(x)}{\partial R_A} = x \quad ; \quad \frac{\partial M_3(x)}{\partial R_C} = x - 2a$

(4) $y_A = \frac{1}{EI}\left[\int_0^a M_1(x)\frac{\partial M_1(x)}{\partial R_A}dx + \int_a^{2a} M_2(x)\frac{\partial M_2(x)}{\partial R_A}dx + \int_{2a}^{3a} M_3(x)\frac{\partial M_3(x)}{\partial R_A}dx\right] = 0$

(5) $y_C = \frac{1}{EI}\left[\int_0^a M_1(x)\underbrace{\frac{\partial M_1(x)}{\partial R_C}}_{=0}dx + \int_a^{2a} M_2(x)\underbrace{\frac{\partial M_2(x)}{\partial R_C}}_{=0}dx + \int_{2a}^{3a} M_3(x)\frac{\partial M_3(x)}{\partial R_C}dx\right] = 0$!

$$④\quad y_A = 0 = \frac{1}{EI}\left[\int_0^a R_A \cdot x \cdot x\,dx + \int_a^{2a}\left[R_A \cdot x + P(x-a) - \frac{q(x-a)^3}{6a}\right]\cdot x\,dx + \right.$$

$$\left. + \int_{2a}^{3a}\left[R_A \cdot x + P(x-a) - \frac{1}{2}qa\left(x - \frac{5}{3}a\right) - M + R_C(x-2a)\right] x\,dx\right]$$

$$⑤\quad y_C = 0 = \frac{1}{EI}\int_{2a}^{3a}\left[R_A \cdot x + P(x-a) - \frac{1}{2}qa\left(x-\frac{5}{3}a\right) - M + R_C(x-a)\right] \cdot$$

$$\cdot (x-2a) = 0$$

$$\left.\begin{matrix} ④ \\ ⑤ \end{matrix}\right\} \Rightarrow R_A, \cancel{R_B}\ R_C \Rightarrow \left.\begin{matrix} \cancel{①} \\ ② \\ ③ \end{matrix}\right\} \Rightarrow R_{DY}, M_D$$

Selection of R_A and ~~R_B~~ R_C reactions as hyperstatic forces is the optimal decision

but keep attention!

it is possible to select for exapmple R_{DY} and M_D as the hyperstatic forces

then

$$④\quad \frac{\partial U}{\partial R_{DY}} = y_D \neq 0$$

$$⑤\quad \frac{\partial U}{\partial M_D} = \vartheta_D \neq 0$$

then, when writing the equations for the bending moments (going from the left side), it is necessary the determine in advance

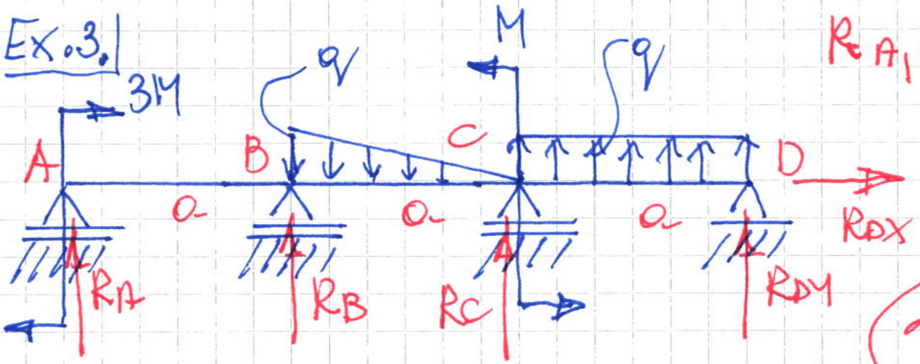
$$R_A = f(q, P, M, R_{DY}, M_D)$$

$$R_C = f^*(q, P, M, R_{DY}, M_D)$$

Homework

$q, a, M = qa^2, EI = \text{const}$

Ex. 3.



$R_A, R_B, R_C, R_{DX}, R_{DY}$ (5 reactions)

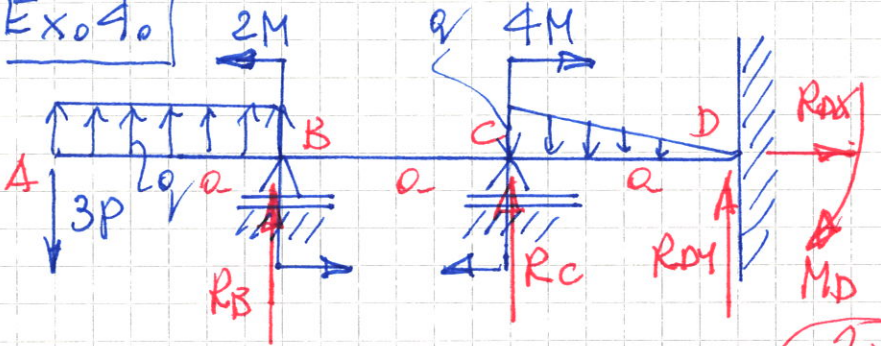
2x hyperstatic beam

Menabre'a-Castigliano's theorem

version a – please select R_A and R_B as hyperstatic forces

version b – please select R_B and R_{DY} as hyperstatic forces

Ex. 4.



$q, a, P = qa, M = qa^2$
$EI = \text{const}$

$R_B, R_C, R_{DX}, R_{DY}, M_D$

2x hyperstatic beam

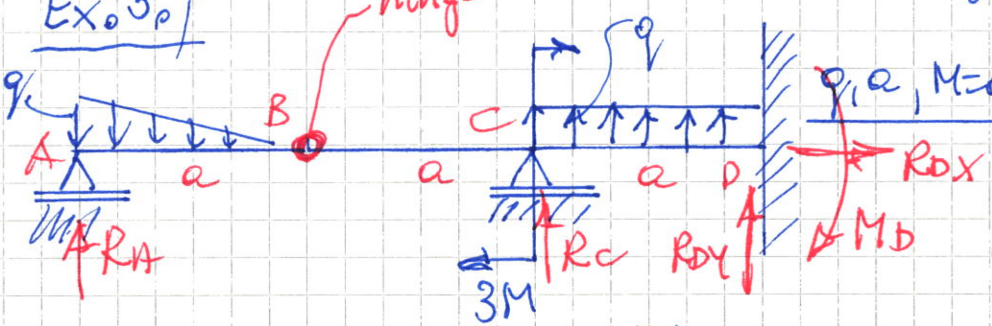
Menabre'a-Castigliano's theorem

version a – please select R_B and R_C as hyperstatic forces

version b – please select R_C and M_D as hyperstatic generalised forces

Ex. 5.

hinge



$q, a, M = qa^2, EI = \text{const}$

$R_A, R_C, R_{DX}, R_{DY}, M_D - 2$

Menabre'a-Castigliano's theorem

2 x hyperstatic beam

please select any hyperstatic force