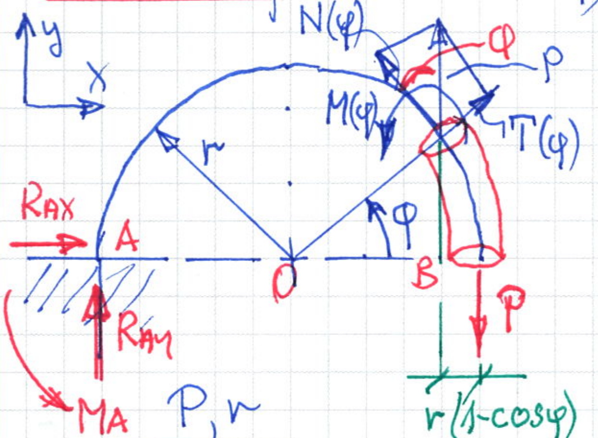
Curved bars; Castigliano's theorem

Exe 1

Internal forces $N(\varphi)$, $T(\varphi)$, $M(\varphi)$



P, r

$N(\varphi), T(\varphi), M(\varphi)$ - ?

The force system on the semicircular beam

(I) static eqs.

(1) $\sum P_{ix} = R_{AX} = 0$

(2) $\sum P_{iy} = R_{AY} - P = 0 \Rightarrow R_{AY} = P$

(3) $\sum M_i^A = -M_A + P \cdot 2r = 0 \Rightarrow$

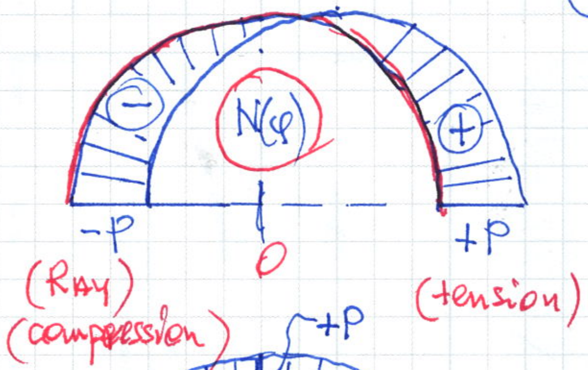
$$M_A = 2 \cdot P \cdot r$$

(II) Internal forces

a) $N(\varphi) = P \cdot \cos\varphi$

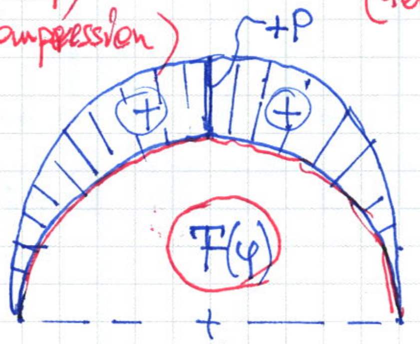
$N(0) = P$, $N\left(\frac{\pi}{2}\right) = 0$

$N(\pi) = -P$



b) $T(\varphi) = P \sin\varphi$

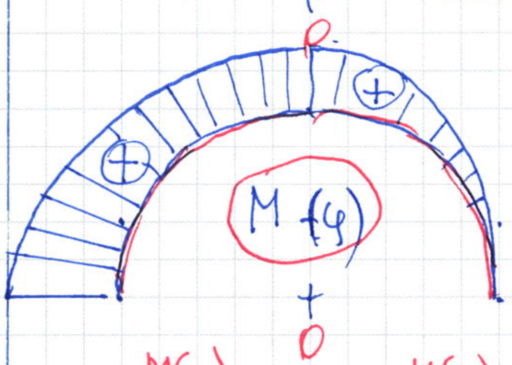
$T(0) = 0$, $T\left(\frac{\pi}{2}\right) = P$, $T(\pi) = 0$



c) $M(\varphi) = P(r - r \cdot \cos\varphi) = P \cdot r(1 - \cos\varphi)$

$M(0) = 0$, $M\left(\frac{\pi}{2}\right) = P \cdot r$

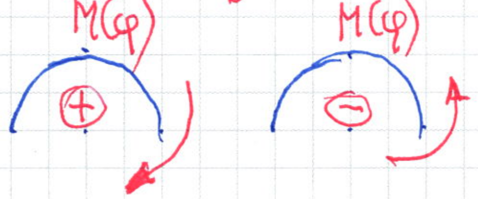
$M(\pi) = 2 \cdot P \cdot r$



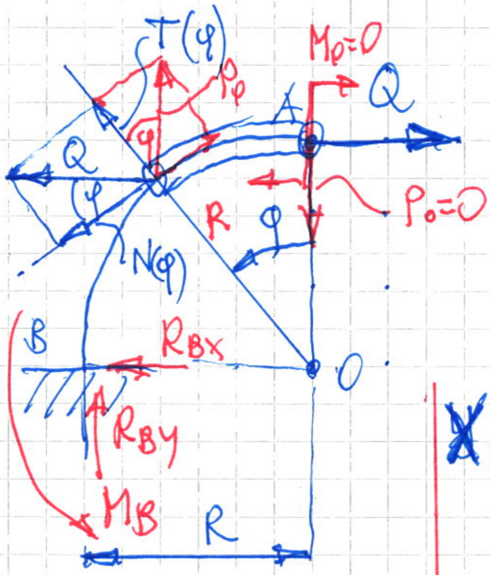
$N(\pi) = -P = R_{AY}$

$T(\pi) = 0 = R_{AX}$

$M(\pi) = 2 \cdot P \cdot r = M_A$



Ex. 2.



$Q,\ R,\ EI=const,\ EA=const,$

$x_A,\ y_A,\ \vartheta_A$ - ?

Castigliano th.

$$x_A=\frac{\partial U}{\partial Q},\quad y_A=\frac{\partial U}{\partial P_0},\quad \vartheta_A=\frac{\partial U}{\partial M_0}$$

$$x_A=\frac{1}{EI}\int_S M(\varphi)\cdot\frac{\partial M(\varphi)}{\partial Q}ds+\frac{1}{EA}\int_S N(\varphi)\cdot\frac{\partial N(\varphi)}{\partial Q}ds$$

$$y_A=\frac{1}{EI}\int_S M(\varphi)\frac{\partial M(\varphi)}{\partial P_0}ds+\frac{1}{EA}\int_S N(\varphi)\frac{\partial N(\varphi)}{\partial P_0}ds$$

$ds=R\cdot d\varphi$

$$\vartheta_A=\frac{1}{EI}\int_S M(\varphi)\frac{\partial M(\varphi)}{\partial M_0}ds+\frac{1}{EA}\int_S N(\varphi)\frac{\partial N(\varphi)}{\partial M_0}ds$$

keep attention!
above, the arbitrary decision, that only bending and normal forces affect elastic energy

$$N(\varphi)=Q\cdot\cos\varphi-P_0\cdot\sin\varphi$$
$$T(\varphi)=Q\cdot\sin\varphi+P_0\cdot\cos\varphi$$
$$M(\varphi)=M_0+P_0\cdot R\cdot\sin\varphi+Q\cdot R(1-\cos\varphi)$$

!

$$\frac{\partial M(\varphi)}{\partial Q}=R(1-\cos\varphi);\quad \frac{\partial M(\varphi)}{\partial P_0}=R\cdot\sin\varphi;\quad \frac{\partial M(\varphi)}{\partial M_0}=1$$

$$\frac{\partial N(\varphi)}{\partial Q}=\cos\varphi;\quad \frac{\partial N(\varphi)}{\partial P_0}=-\sin\varphi;\quad \frac{\partial N(\varphi)}{\partial M_0}=0$$

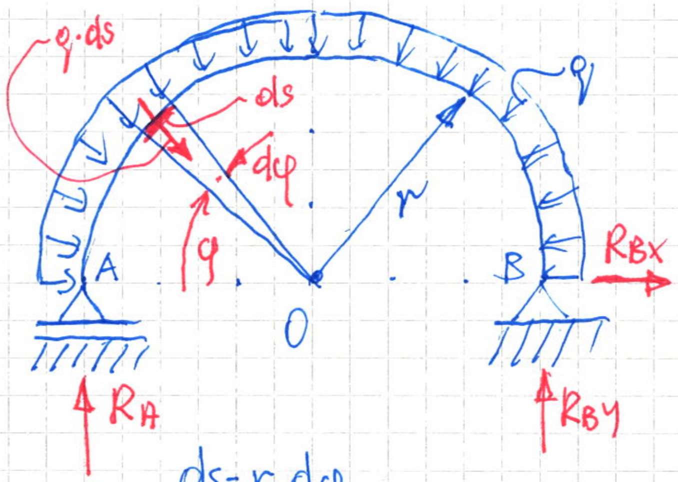
$$\frac{\partial T(\varphi)}{\partial Q}=\sin\varphi;\quad \frac{\partial T(\varphi)}{\partial P_0}=\cos\varphi,\quad \frac{\partial T(\varphi)}{\partial M_0}=0$$

$$x_A=\frac{1}{EI}\int_0^{\frac{\pi}{2}}[\underbrace{M_0}_{=0}+\underbrace{P_0}_{=0}\cdot R\sin\varphi+Q\cdot R(1-\cos\varphi)]\,R(1-\cos\varphi)\cdot R\,d\varphi+$$

$$+\frac{1}{EA}\int_0^{\frac{\pi}{2}}[Q\cdot\cos\varphi-\underbrace{P_0}_{=0}\sin\varphi]\cdot\cos\varphi\cdot R\,d\varphi=\frac{1}{EI}\int_0^{\frac{\pi}{2}}Q\cdot R^3(1-\cos\varphi)^2d\varphi$$

$$+\frac{1}{EA}\int_0^{\frac{\pi}{2}}Q\cdot\cos^2\varphi\cdot R\cdot d\varphi=\ ?$$

Ex.3.) Curved bars, continuous load



$ds = r \cdot d\varphi$

$\underline{q,\ r}$
reactions - 2

Ⓣ static eqs.

① $\sum P_{ix} = \int_0^{\pi} q \cdot \cos\varphi \underbrace{ds}_{r \cdot d\varphi} + R_{Bx} = 0$

$= q \cdot r \int_0^{\pi} \cos\varphi \, d\varphi + R_{Bx} = 0$

$q r \sin\varphi \Big|_0^{\pi} + R_{Bx} = 0$

$q r (0 - 0) + R_{Bx} = 0 \Rightarrow \underline{R_{Bx} = 0}$

② $\sum P_{iy} = R_A + R_{By} - \int_0^{\pi} q r \sin\varphi \, d\varphi = 0$

$R_A + R_{By} + q r \cos\varphi \Big|_0^{\pi} = 0$

$R_A + R_{By} + q r (-1 - 1) = 0$

$R_A + R_{By} - 2 q r = 0$

③ $\sum M_i^{O} = R_A \cdot r - R_{By} \cdot r = 0$

$R_A = R_{By}$

→ ② $2 R_A - 2 q r = 0 \Rightarrow$

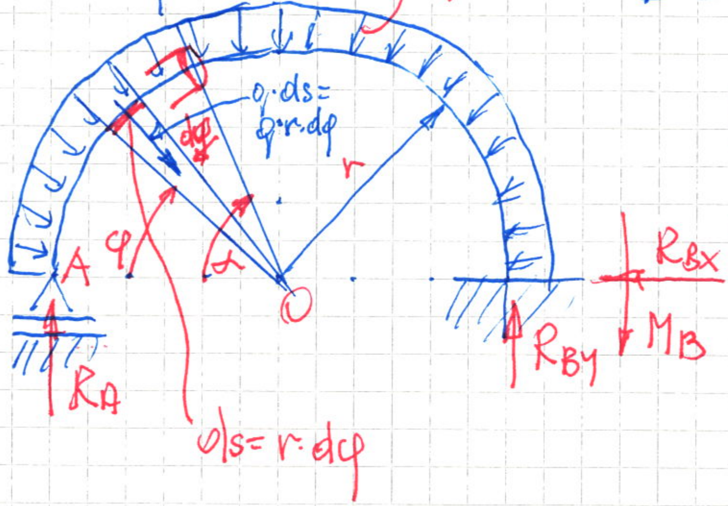
$\Rightarrow \underline{R_A = R_{By} = q \cdot r}$

Ex. 4.

$q, r, EI = const, EA = const$

reactions - ?

Menabrea - Castigliano th.



I static eqs.

$$① \sum P_{ix} = \int_0^{\pi} q \cdot \cos\varphi \cdot r \cdot d\varphi + R_{Bx} = 0$$

$$② \sum P_{iy} = R_A + R_{By} - \int_0^{\pi} q \cdot r \cdot \sin\varphi \, d\varphi = 0$$

$$③ \sum M_i^0 = R_A \cdot r - R_{By} \cdot r + M_B = 0$$

4 reactions - 3 st. eqs. ⇒

⇒ 1x hyperstatic curved bar

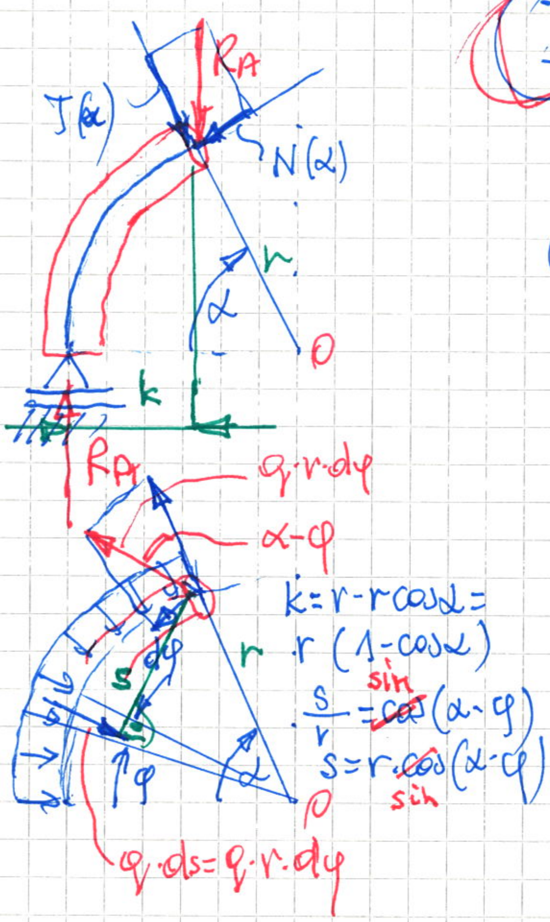
R_A - hyperstatic reaction

④ geometrical eq.

$$\frac{\partial U}{\partial R_A} = y_A = 0$$

Menabrea - Castigliano th.

$$y_A = \frac{1}{EI} \int_0^{\pi} M(\alpha) \cdot \frac{\partial M(\alpha)}{\partial R_A} r \cdot d\varphi + \frac{1}{EA} \int_0^{\pi} N(\alpha) \frac{\partial N(\alpha)}{\partial R_A} r d\varphi = 0$$



II Internal forces

$$N(\alpha) = -R_A \cdot \cos\alpha - \int_0^{\alpha} q \cdot r \sin(\alpha - \varphi) \, d\varphi$$

$$\frac{\partial N(\alpha)}{\partial R_A} = -\cos\alpha$$

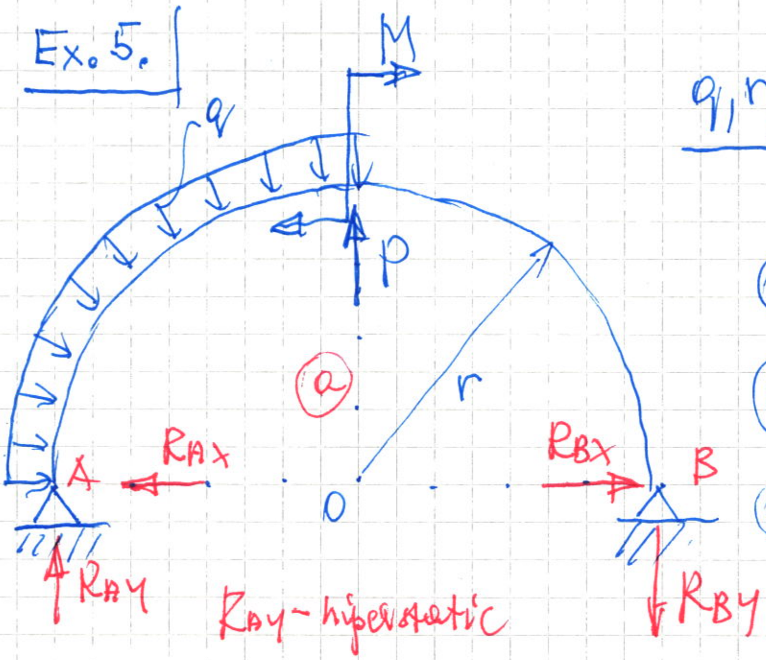
$$M(\alpha) = -R_A \cdot k + \int_0^{\alpha} q \cdot r \cdot s \cdot d\varphi =$$

$$= -R_A \cdot r \cdot (1 - \cos\alpha) + \int_0^{\alpha} q \cdot r \cdot r \cdot \cancel{\cos}\,\sin(\alpha - \varphi) \, d\varphi =$$

$$= -R_A \cdot r (1 - \cos\alpha) + q r^2 \int_0^{\alpha} \cancel{\cos}\,\sin(\alpha - \varphi) \, d\varphi$$

$$\frac{\partial M(\alpha)}{\partial R_A} = -r(1 - \cos\alpha)$$

Ex. 5.



$q, r, P = q \cdot r, M = q \cdot r^2$

reactions - ?

1. Menabrea - Castigliano
2. Maxwell - Mohr
3. Force Method

for a

I static eqs

$$\left.\begin{array}{l} (1)\ \sum P_{ix} = 0 \\ (2)\ \sum P_{iy} = 0 \\ (3)\ \sum M_i^{O} = 0 \end{array}\right\} \Rightarrow$$

4 reactions - 3 st. eqs $\Rightarrow$
$\Rightarrow$ 1 × hyperstatic curved bar

Menabrea - Castigliano th.

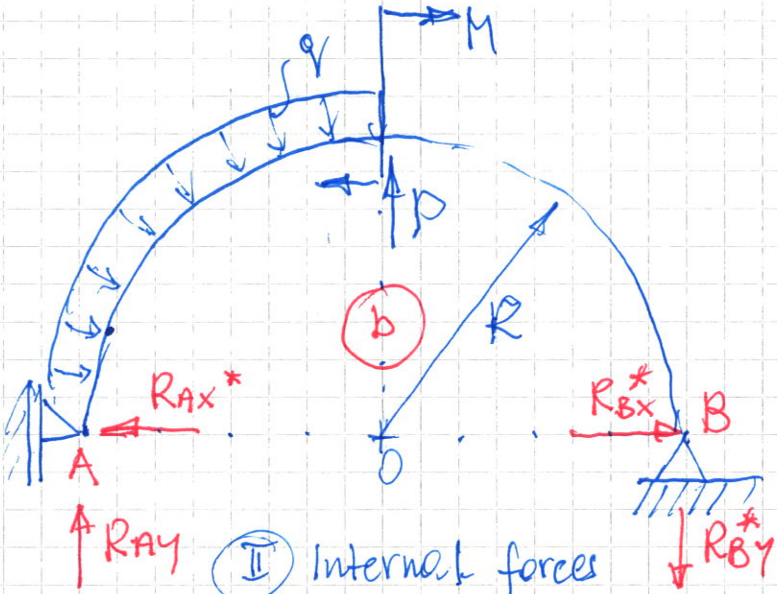
$$(4)\quad \frac{\partial U}{\partial R_{Ay}} = y_A = 0$$

$0 \le \varphi_1 \le \frac{\pi}{2}$	$\frac{\pi}{2} \le \varphi_2 \le \pi$
$M_1(\varphi) =$	$M_2(\varphi) = \ldots$
$N_1(\varphi) =$	$N_2(\varphi) =$

$$y_A = 0 = \frac{1}{EI}\left[\int_0^{\frac{\pi}{2}} M_1(\varphi) \cdot \frac{\partial M_1(\varphi)}{\partial R_{Ay}} r\,d\varphi + \int_{\frac{\pi}{2}}^{\pi} M_2(\varphi) \frac{\partial M_2(\varphi)}{\partial R_{Ay}} r\,d\varphi\right] +$$

$$+ \frac{1}{EA}\left[\int_0^{\frac{\pi}{2}} N_1(\varphi) \cdot \frac{\partial N_1(\varphi)}{\partial R_{Ay}} r\,d\varphi + \int_{\frac{\pi}{2}}^{\pi} N_2(\varphi) \cdot \frac{\partial N_2(\varphi)}{\partial R_{Ay}} r\,d\varphi\right] = 0 \Rightarrow$$

$\Rightarrow R_{Ay} \Rightarrow$ (1), (2), (3) $\Rightarrow R_{Ax}, R_{Bx}, R_{By}$



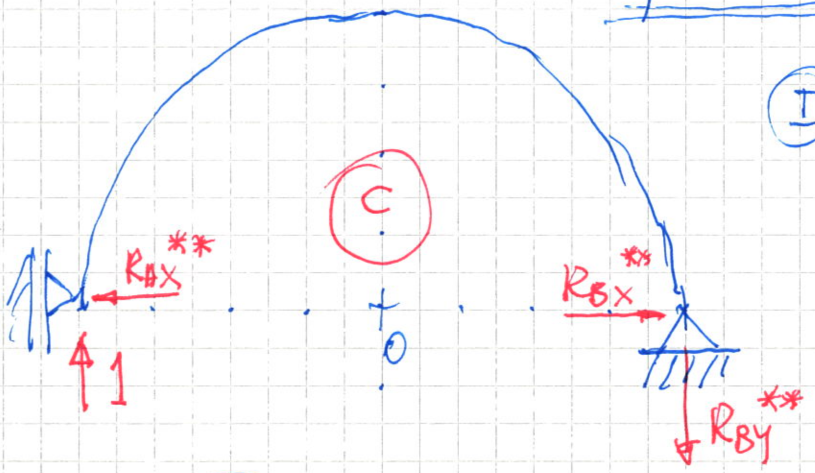
Maxwell-Mohr th.

Ⓘ static eqs.

$$\left.\begin{array}{l} (1)\ \sum P_{ix}=0 \\ (2)\ \sum P_{iy}=0 \\ (3)\ \sum M_i^0=0 \end{array}\right\} \Rightarrow R_{Ax}^*, R_{Bx}^*, R_{By}^*$$

Ⅱ Internal forces

$0 \le \varphi_1 \le \frac{\pi}{2}$	$\frac{\pi}{2}\ \varphi_2 \le \pi$
$M_1^*(\varphi) =$	$M_2^*(\varphi) =$
$N_1^*(\varphi) =$	$N_2^*(\varphi) =$



Ⓘ static eqs.

$$\left.\begin{array}{l} (1)\ \sum P_{ix}=0 \\ (2)\ \sum P_{iy}=0 \\ (3)\ \sum M_i^0=0 \end{array}\right\} \Rightarrow R_{Ax}^{**}, R_{Bx}^{**}, R_{By}^{**}$$

Ⅱ Internal forces

$ds = R \cdot d\varphi$

$0 \le \varphi_{1/2} \le \pi$

$M_{1/2}^0(\varphi) =$

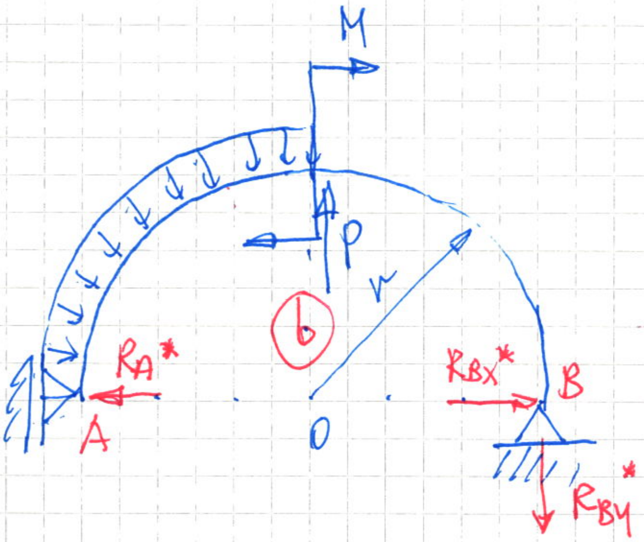
$N_{1/2}^0(\varphi) = 0$

$$(4)\ y_A = \frac{1}{EI}\left[\int_0^{\frac{\pi}{2}} M_1^*(\varphi)\cdot M_{1/2}^0(\varphi)\, R\,d\varphi + \int_{\frac{\pi}{2}}^{\pi} M_2^*(\varphi)\, M_{1/2}^0(\varphi)\, R\,d\varphi\right] +$$

$$+ \frac{1}{EA}\left[\int_0^{\frac{\pi}{2}} N_1^*(\varphi)\cdot N_{1/2}^0(\varphi)\, R\,d\varphi + \int_{\frac{\pi}{2}}^{\pi} N_2^*(\varphi)\cdot N_{1/2}^0(\varphi)\, R\,d\varphi\right] = 0\ !$$

$\Rightarrow R_{Ay} \Rightarrow (1), (2), (3) \Rightarrow R_{Ax}, R_{Bx}, R_{By}$

Force Method

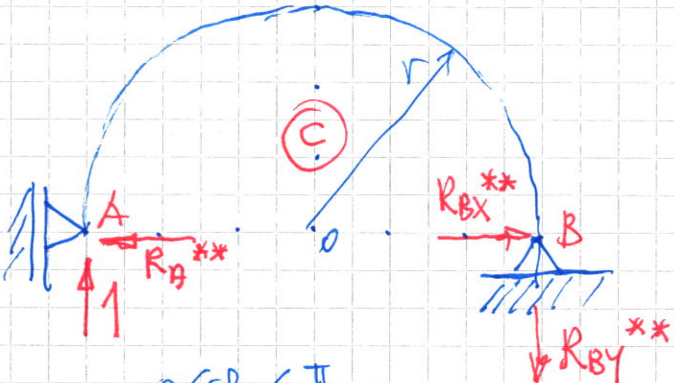


(I) static eqs.

$$\left.\begin{array}{l} (1)\ \Sigma P_{ix} = 0 \\ (2)\ \Sigma P_{iy} = 0 \\ (3)\ \Sigma M_i^{b} = 0 \end{array}\right\} \Rightarrow R_A^*,\ R_{Bx}^*,\ R_{By}^*$$

(II) Internal forces

$0 \le \varphi_1 \le \frac{\pi}{2}$	$\frac{\pi}{2} \le \varphi_2 \le \pi$
$M_1^*(\varphi) =$ $N_1^*(\varphi) =$	$M_2^*(\varphi) =$ $N_2^*(\varphi) =$



(I) static eqs.

$$\left.\begin{array}{l} (1)\ \Sigma P_{ix} = 0 \\ (2)\ \Sigma P_{iy} = 0 \\ (3)\ \Sigma M_i = 0 \end{array}\right\} R_A^{**},\ R_{Bx}^{**},\ R_{By}^{**}$$

(II) Internal forces

$0 \le \varphi_1 \le \frac{\pi}{2}$
$M_1^{**}(\varphi) =$
$N_1^{**}(\varphi) =$

$\frac{\pi}{2} \le \varphi_2 \le \pi$
$M_2^{**}(\varphi) =$
$N_2^{**}(\varphi) =$

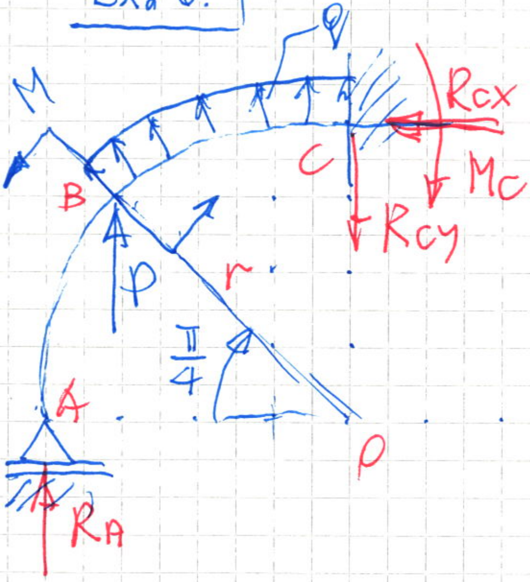
$R_{AY} = X_1 \Rightarrow$ hyperstatic

(4) $X_1 \delta_{11} + \Delta_{1P} = 0$

$$\left.\begin{array}{l} \Delta_{1P} \Rightarrow (b) + (c) \\ \delta_{11} \Rightarrow (c) + (c) \end{array}\right\} \Rightarrow X_1 = R_{AY} \Rightarrow (1), (2), (3)$$

for (a) $\Rightarrow R_{AX}, R_{BX}, R_{BY}$

Ex. 6.



$q, r, P = q \cdot r, M = qr^2$

$EI = const, EA = const$

reactions - ?

1. Menabrea - Castigliano th.
2. Maxwell - Mohr method
3. Force method