Maxwell-Mohr method, Vereschagin rule

Graph multiplication method presents most effective way for computation of any displacement (linear, angular, mutual, etc) of bending structures.
The advantage of this method is that the integration procedure according to Maxwell-Mohr integral is replaced by elementary algebraic procedure on two bending moment diagrams in the:
- actual state and
- unit state,

This method was developed by Russian engineer Vereshchagin (1925) and is often referred as the Vereschagin rule

Let us consider some portion AB which is a part of a bending structure (actual state). The bending stiffness, EI, within of this portion is constant.
The bending moment diagram for this portion in actual state is $M_i(x)$
The bending moment diagram for the same structure but for elementary loading (unit state) is $M_i^*(x)$
In general case, a bending moment diagram $M_i(x)$ in the actual state is bounded by curve, but for unit state it may be bounded by straight line (if the structure is subjected to concentrated elementary (1) force or moment.

$$f = \ldots \underbrace{\frac{1}{EI} \sum_i \int_{(l_i)} M_i(x) \cdot M_i^*(x)\, dx}_{\text{only for bending (Maxwell-Mohr method)}}$$

$M_i^*(x) = a \cdot x + b$ (straight line)

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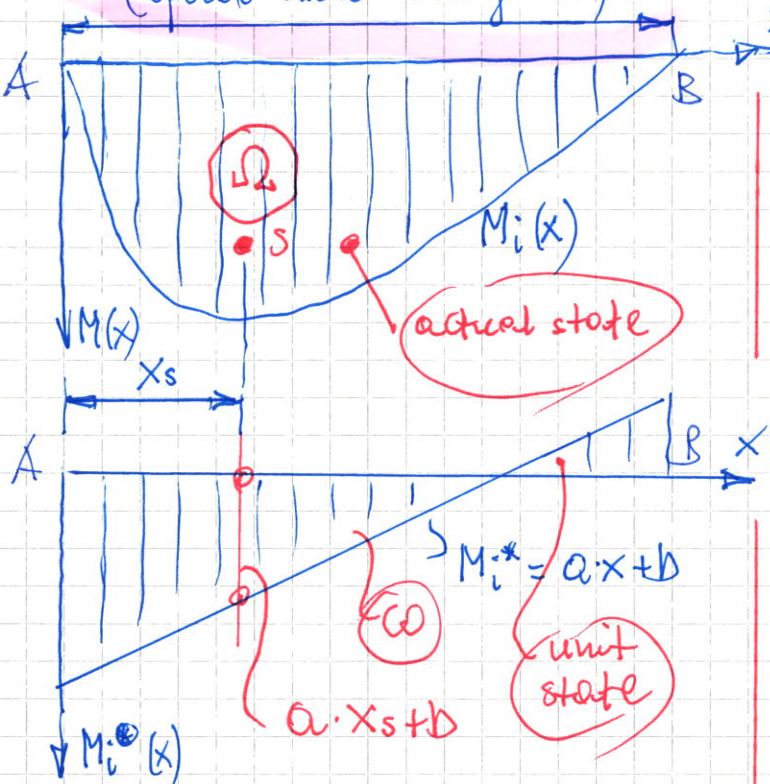
Vereshchagin procedure (sposób Wereszczagina)



S – centroid of $M_i(x)$ graph
Ω – area of bending moment diagram
x_s – coordinate of its centroid
$a\cdot x_s+b$ – ordinate from the bending moment (unit)

$$\int_0^l M_i(x)\cdot M_i^*(x)\,dx =$$

$$\int_0^l M_i(x)\cdot(a\cdot x+b)\,dx =$$

$$\int_0^l M_i(x)\cdot a\cdot x\cdot dx + b\int_0^l M_i(x)\,dx$$

$$= a\underbrace{\int_0^l M_i(x)\cdot x\cdot dx}_{\text{static moment}} + b\underbrace{\int_0^l M_i(x)\,dx}_{\text{total area} - \Omega} =$$

$$\int_0^l M_i(x)\,dx = \Omega$$

$$\int_0^l M_i(x)\cdot x\,dx = \Omega\cdot x_s \quad \Rightarrow$$

$$\int_0^l M_i(x)\cdot M_i^*(x)\,dx =$$

$$a\cdot\Omega\cdot x_s + b\cdot\Omega$$

$$\Omega\,(a\cdot x_s+b)$$

$$\int_0^l M_i(x)\cdot M_i^*(x)\,dx = \Omega\,(a\cdot x_s+b)$$

procedure is called the "multiplication" of two graphs

$$\begin{matrix} y(x) \\ \vartheta(x) \end{matrix} = \frac{1}{EJ}\left[\int_0^l M_A(x)\cdot M_i^*(x)\,dx + \int_{l_i}^{l_{i+1}} M_i(x)\cdot M_i^*(x)\,dx\right] =$$

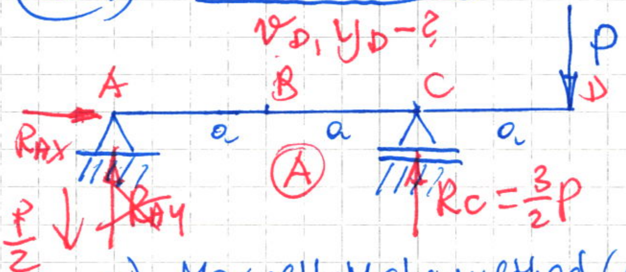
$$= \frac{1}{EJ}\sum_{i=1}^{n}\left[\Omega_i\,(a_i x_{si}+b_i)\right]$$

$-1a$ ←

Result of multiplication = equals to the area Ω of the bending moment diagram in actual state multiplied by ordinate $(a x_s+b)$ from the unit bending moment M_i^*, which is located under the centroid of $M_i(x)$ diagram

Ex. 1. $P, a, EI = \text{const}$

$v_D, y_D - ?$



a) Maxwell-Mohr method (analyt.)

b) M-M, Vereshchagin procedure

Ⓘ static eqs.

① $\sum P_{ix} = R_{Ax} = 0$

② $\sum P_{iy} = R_{Ay} + R_C - P = 0$

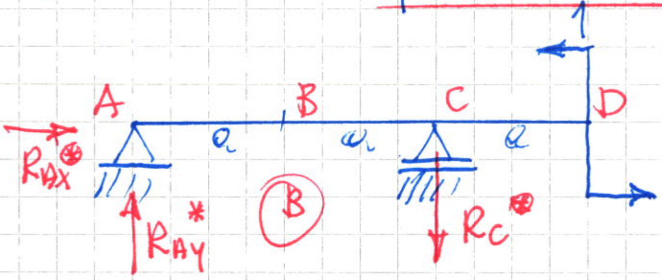
③ $\sum M_i^C = R_{Ay} \cdot 2a + P \cdot a = 0 \;|: a$

$R_{Ay} = -\frac{P}{2}$

② $-\frac{P}{2} + R_C - P = 0 \quad R_C = \frac{3}{2}P$

Ⓘ Bending m.

$0 \le x_1 \le a$	$a \le x_2 \le 2a$	$2a \le x_3 \le 3a$
$M_1(x) = -\frac{P}{2} \cdot x$	$M_2(x) = -\frac{P}{2} \cdot x$	$M_3(x) = -\frac{P}{2} \cdot x + \frac{3}{2}P(x-2a) = -\frac{Px}{2} + \frac{3}{2}P \cdot x - 3Pa = \frac{2Px}{2} - 3Pa = P \cdot x - 3Pa$



Ⓘ static eqs.

① $\sum P_{ix} = R_{Ax}^* = 0$

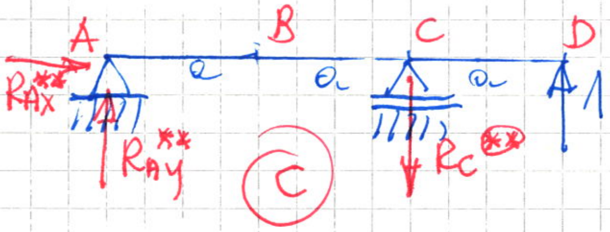
② $\sum P_{iy} = R_{Ay}^* - R_C^* = 0 \Rightarrow R_{Ay}^* = R_C^*$

③ $\sum M_i^C = R_{Ay}^* \cdot 2a - 1 = 0$

$R_{Ay}^* = \frac{1}{2a} = R_C^*$

Ⓘ Bending moments

$0 \le x_1 \le a$	$a \le x_2 \le 2a$	$2a \le x_3 \le 3a$
$M_1^*(x) = R_{Ay}^* \cdot x = \frac{x}{2a}$	$M_2^*(x) = R_{Ay}^* \cdot x = \frac{x}{2a}$	$M_3^*(x) = R_{Ay}^* \cdot x - R_C^*(x-2a) = \frac{x}{2a} - \frac{1}{2a}(x-2a) = \frac{x}{2a} - \frac{x}{2a} + 1 = 1$



Ⓘ static eqs.

① $\sum P_{ix} = R_{Ax}^{**} = 0$

② $\sum P_{iy} = R_{Ay}^{**} - R_C^{**} + 1 = 0$

③ $\sum M_i^C = R_{Ay}^{**} \cdot 2a - 1 \cdot a = 0 \quad R_{Ay}^{**} = \frac{1}{2}$

from ② $\frac{1}{2} - R_C^{**} + 1 = 0 \quad R_C^{**} = \frac{3}{2}$

Ⓘ Bending moments

$0 \le x_1 \le a$	$a \le x_2 \le 2a$	$2a \le x_3 \le 3a$
$M_1^{**}(x) = R_{Ay}^{**} \cdot x = \frac{x}{2}$	$M_2^{**}(x) = \frac{x}{2}$	$M_3^{**}(x) = \frac{x}{2} - \frac{3}{2}(x-2a) = -x + 3a$

Metoda M-M (analyt.)

A + B

$$\vartheta_{\cancel{B}D} = \frac{1}{EJ}\left[\int_0^a \left(-\frac{P}{2}\cdot x\right)\cdot\frac{x}{2a}\,dx + \int_a^{2a}\left(-\frac{P}{2}\cdot x\right)\cdot\frac{x}{2a}\,dx + \int_{2a}^{3a}(P\cdot x-3Pa)\cdot 1\,dx\right] =$$

$$=\frac{1}{EJ}\left[\int_0^a -\frac{Px^2}{4a}\,dx + \int_a^{2a} -\frac{Px^2}{4a}\,dx + \int_{2a}^{3a}(Px-3Pa)\,dx\right]=$$

$$=\frac{1}{EJ}\left[-\frac{Px^3}{12a}\Big|_0^a - \frac{Px^3}{12a}\Big|_a^{2a} + \left(P\frac{x^2}{2}-3Pax\right)\Big|_{2a}^{3a}\right]=$$

$$=\frac{1}{EJ}\left[-\frac{Pa^3}{12a} - \frac{P}{12a}(8a^3-a^3) + \left(\frac{P}{2}9a^2 - 3Pa\cdot 3a - \frac{P}{2}4a^2 + 3Pa\cdot 2a\right)\right]=$$

$$=\frac{1}{EJ}\left[-\frac{Pa^2}{12} - \cancel{\frac{2}{3}Pa^2} - \frac{7}{12}Pa^2 + \left(\frac{9Pa^2}{2} - 9Pa^2 - \underbrace{2Pa^2+6Pa^2}_{4Pa^2}\right)\right]=$$

$$=\frac{1}{EJ}\left[-\frac{2}{3}Pa^2 - \frac{Pa^2}{2}\right] = -\frac{7Pa^2}{6EJ}$$

A + C

$$y_{\cancel{A}D} = \frac{1}{EJ}\left[\int_0^a \left(-\frac{P}{2}x\right)\frac{x}{2}\,dx + \int_a^{2a}\left(-\frac{P}{2}x\right)\frac{x}{2}\,dx + \int_{2a}^{3a}(P\cdot x-3Pa)\cdot 1\,dx\right] =$$

$$=\frac{1}{EJ}\left[\int_0^a -\frac{Px^2}{4}\,dx + \int_a^{2a} -\frac{Px^2}{4}\,dx + \int_{2a}^{3a}(Px-3Pa)\,dx\right]=$$

$$=\frac{1}{EJ}\left[-\frac{Px^3}{12}\Big|_0^a - \frac{Px^3}{12}\Big|_a^{2a} + \left(P\frac{x^2}{2}-3Pa\cdot x\right)\Big|_{2a}^{3a}\right]=$$

$$=\frac{1}{EJ}\left[-\frac{Pa^3}{12} - \frac{P}{12}(8a^3-a^3) + \left(\frac{P}{2}9a^2 - 3Pa\cdot 3a - \frac{P}{2}\cdot 4a^2 + 3Pa\cdot 2a\right)\right]=$$

$$=\frac{1}{EJ}\left[-\frac{Pa^3}{12} - \frac{7Pa^3}{12} + Pa^3\left(\frac{9}{2} - \right.\right.$$

$$y_C = \frac{1}{EJ}\left[\int_0^a \left(-\frac{P}{2}x\right)\frac{x}{2}\,dx + \int_a^{2a}\left(-\frac{P}{2}\cdot x\right)\cdot\frac{x}{2}\,dx + \int_{2a}^{3a}(P\cdot x-3Pa)(-x+3a)\,dx\right] =$$

$$=\frac{1}{EJ}\left[\int_0^a -\frac{Px^2}{4}\,dx + \int_a^{2a} -\frac{Px^2}{4}\,dx + \int_{2a}^{3a}\left(-Px^2+3Pxa+3Pax-9Pa^2\right)dx\right]=$$

$$y^{D}_{\cancel{C}} = \frac{1}{EI}\left[-\frac{Px^3}{12}\Big|_0^{a} - \frac{Px^3}{12}\Big|_a^{2a} + \left(-\frac{Px^3}{3} + 3Pa\frac{x^2}{2} + 3Pa\frac{x^2}{2} - 9Pa^2x\right)\Big|_{2a}^{3a}\right] =$$

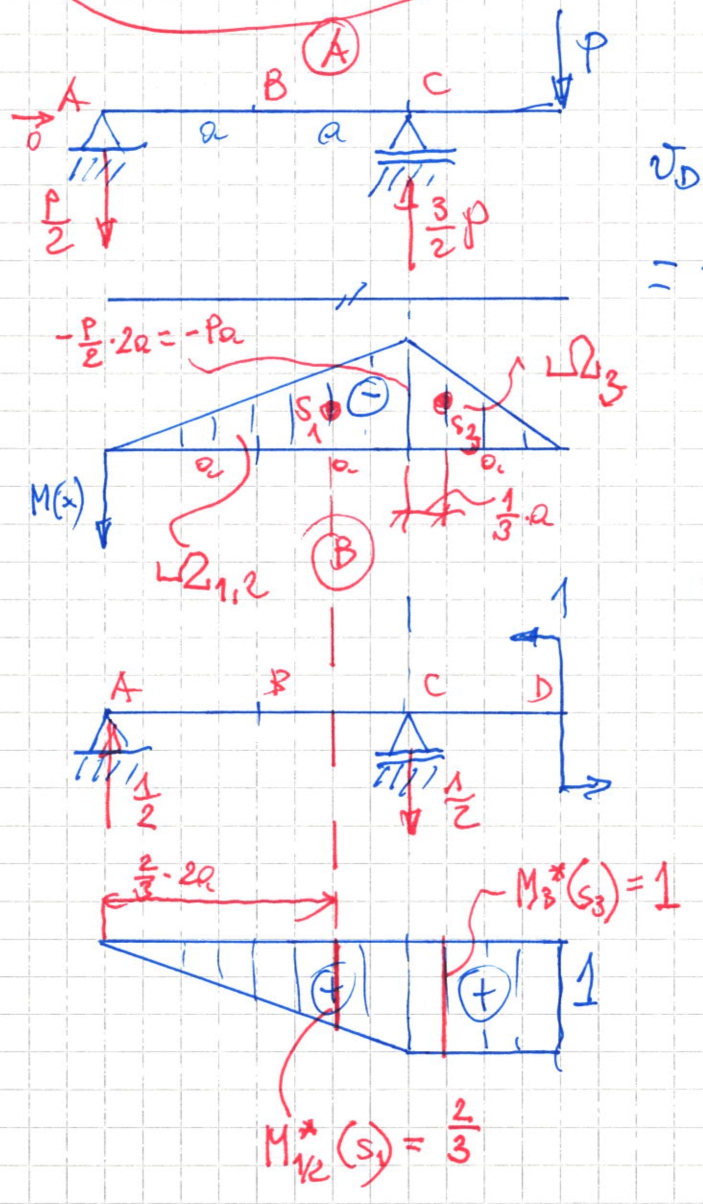
$$= \frac{1}{EI}\Bigg[-\frac{Pa^3}{12} - \frac{P}{12}\left(8a^3 - a^3\right) + \Big(-\frac{P}{3}\cdot 27a^3 + \frac{3Pa}{2}\cdot 9a^2 + \frac{3Pa}{2}\cdot 9a^2 - 9Pa^2\cdot 3a$$

$$+ \frac{P}{3}8a^3 - \frac{3Pa}{2}4a^2 - \frac{3Pa}{2}4a^2 + 9Pa^2\cdot 2a\Big)\Bigg] =$$

$$= \frac{1}{EI}\left[-\frac{2}{3}Pa^3 + Pa^3\left(-\cancel{9} + \cancel{\frac{27}{2}} + \cancel{\frac{27}{2}} - \cancel{27} + \frac{8}{3} - 6 - 6 + \cancel{18}\right)\right] =$$

$$= \frac{1}{EI}\left(-\frac{2}{3}Pa^3 - 3Pa^3 + \frac{8}{3}Pa^3\right) = -\frac{Pa^3}{EI}$$

V. procedure



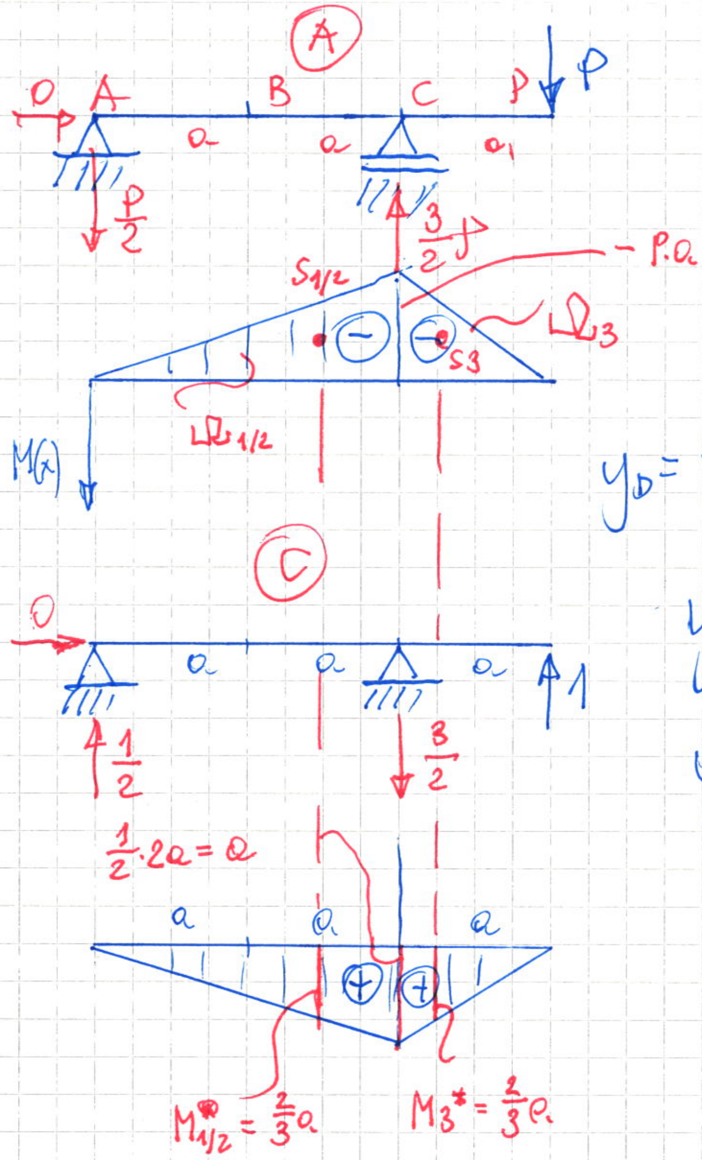
$$v_D = \frac{1}{EI}\left[\Omega_{1/2}\cdot\frac{2}{3} + \Omega_3\cdot 1\right] =$$

$$= \frac{1}{EI}\left[\left(-Pa^2\right)\cdot\frac{2}{3} + \left(-\frac{Pa^2}{2}\right)\cdot 1\right] = -\frac{Pa^2}{EI}\left(\frac{2}{3} + \frac{1}{2}\right) =$$

$$= -\frac{7}{6}\frac{Pa^2}{EI}$$

$$\Omega_{1/2} = \frac{1}{2}\cdot 2a\cdot\left(-P\cdot a\right) = -Pa^2$$

$$\Omega_3 = \frac{1}{2}a\cdot\left(-Pa\right) = -\frac{Pa^2}{2}$$



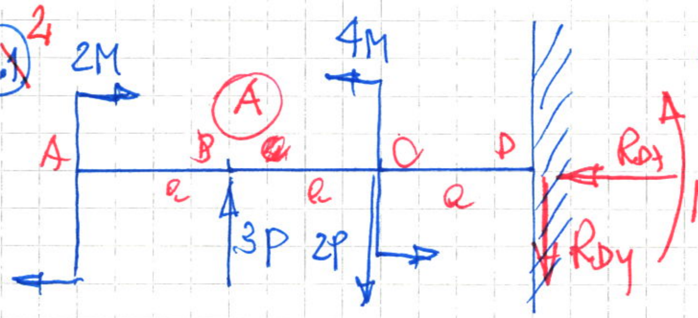
$$y_D = \frac{1}{EI}\left[\Omega_{1/2}\cdot\frac{2}{3}a + \Omega_3\cdot\frac{2}{3}a\right] =$$

$$y_D = \frac{1}{EI}\left[\left(-Pa^2\right)\cdot\frac{2}{3}a + \left(-\frac{Pa^2}{2}\right)\cdot\frac{2}{3}a\right]$$

$$y_D = \frac{-Pa^2}{EI}\left(\frac{2}{3}a + \frac{1}{2}\cdot\frac{2}{3}a\right) =$$

$$y_D = \frac{-Pa^2}{EI}\cdot\cancel{\frac{2}{3}}a\cdot\cancel{\frac{3}{2}} = -\frac{Pa^3}{EI}$$

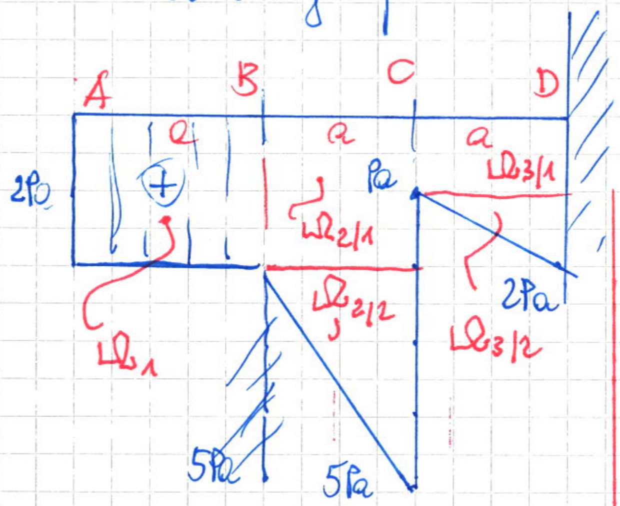
Ex. 2



$P, a, M = Pa, EI = const$

$\vartheta_B, y_C - ?$

Vereshchagin procedure



I Static eqs.

① $\sum P_{ix} = -R_{Dx} = 0$

② $\sum P_{iy} = 3P - 2P - R_{Dy} = 0$

$P - R_{Dy} = 0 \Rightarrow R_{Dy} = P$

③ $\sum M_{iD} = 2M + 3P \cdot 2a - 2Pa - 4M - M_D = 0$

$2Pa + 6Pa - 2Pa - 4Pa - M_D = 0$

$M_D = 2Pa$

II Bending moments

$0 \le x_1 \le a$

$M_1(x) = 2M = 2Pa$

$a \le x_2 \le 2a$

$M_2(x) = 2M + 3P(x-a)$

$M_2(a) = 2Pa + 3P(a-a) = 2Pa$

$M_2(2a) = 2Pa + 3P \cdot a = 5Pa$

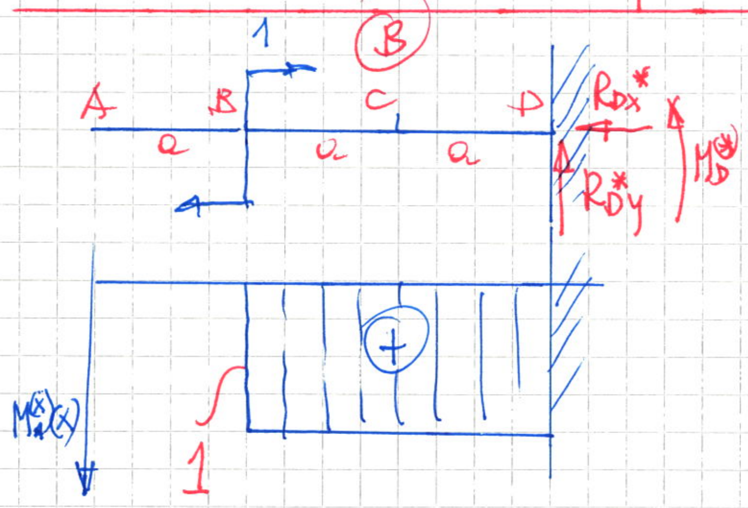
$2a \le x_3 \le 3a$

$M_3(2a) = 5Pa - 4Pa = Pa$

$M_3(3a) = 2M + 3P \cdot 2a - 2P \cdot a - 4Pa =$

$2Pa + 6Pa - 2Pa - 4Pa = 2Pa$

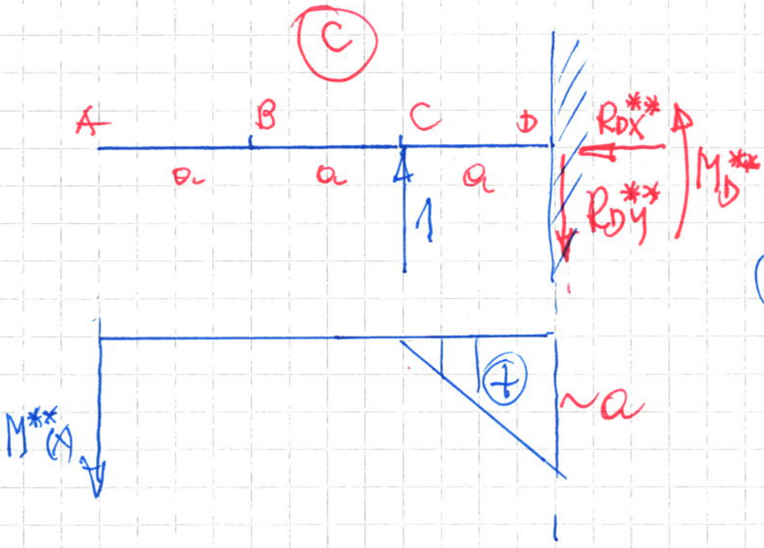
B



$M_1^*(x) = 0$

$M_2^*(x) = 1$

$M_3^*(x) = 1$



I) Static eqs.

(1) $\sum P_{ix} = R_{Dx} = 0$

(2) $\sum P_{iy} = 1 - R_{Dy}^{**} = 0 \Rightarrow R_{Dy}^{**} = 1$

(3) $\sum M_i^D = 1 \cdot a - M_D^{**} = 0 \quad M_D^{**} = 1a = a$

$\Omega_1 = 2Pa \cdot a = 2Pa^2$

$\Omega_{2/1} = 2Pa^2$

$\Omega_{2/2} = \frac{1}{2} a \cdot 3Pa = \frac{3}{2} Pa^2$

$\Omega_{3/1} = Pa^2$

$\Omega_{3/2} = \frac{1}{2} Pa^2$

A + B

$$\vartheta_B = \frac{1}{EI}\left[\Omega_1 \cdot 0 + \Omega_{2/1} \cdot 1 + \Omega_{2/2} \cdot 1 + \Omega_{3/1} \cdot 1 + \Omega_{3/2} \cdot 1\right]$$

$$\vartheta_B = \frac{1}{EI}\left[2Pa^2 \cdot 1 + \frac{3}{2} Pa^2 \cdot 1 + Pa^2 \cdot 1 + \frac{1}{2} Pa^2 \cdot 1\right] = \frac{5Pa^2}{EI}$$

$$y_C = \frac{1}{EI}\left[0 + 0 + \Omega_{3/1} \cdot \frac{a}{2} + \Omega_{3/2} \cdot \frac{2}{3} a\right] =$$

$$y_C = \frac{1}{EI}\left[Pa^2 \cdot \frac{a}{2} + \frac{Pa^2}{2} \cdot \frac{2}{3} a\right] = \frac{1}{EI}\left[\frac{Pa^3}{2} + \frac{Pa^3}{3}\right]$$

$$y_C = \frac{5}{6EI} Pa^3$$