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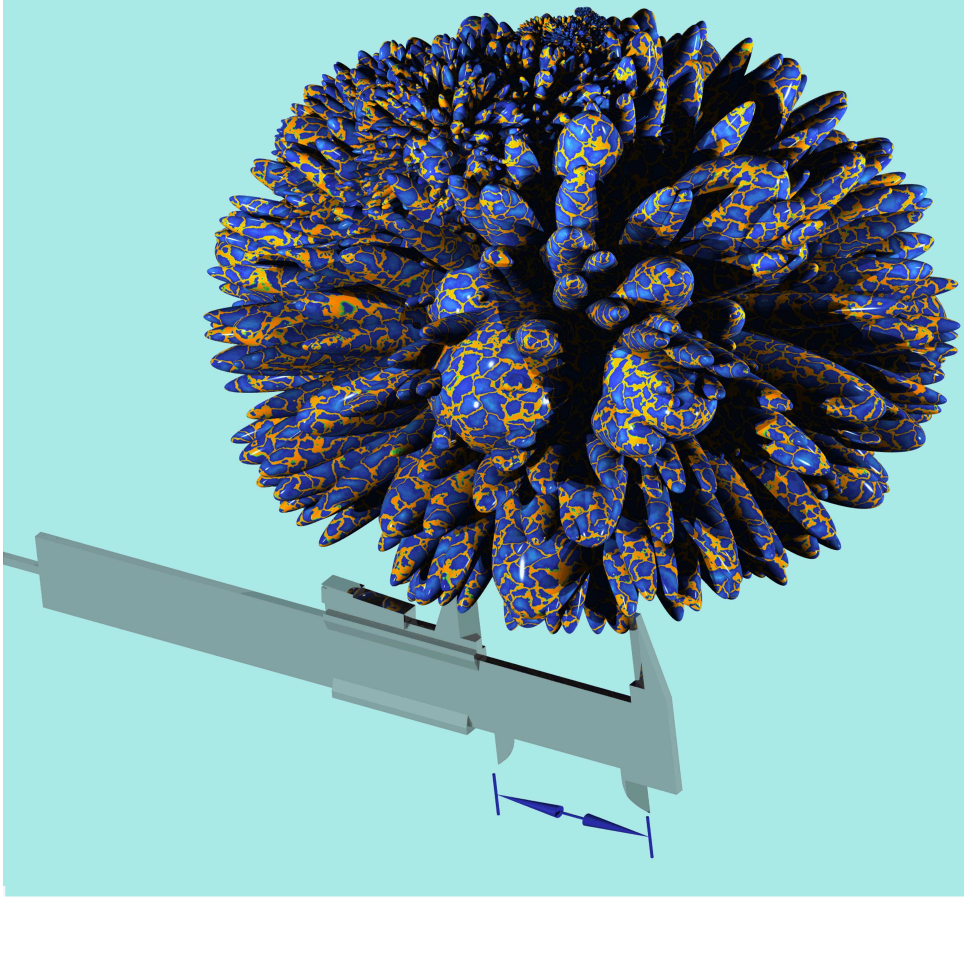
# Measurements, Dimensions, Invariant Models and Fractals



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# Preface

The description of technical, physical and economic processes requires, on the basis of empirical data, the utilization of algorithmic procedures developed within many branches of sciences. They include theories of identification, planning of experiments and dimensional analysis. The dimensional analysis, relating to studies of mathematical models on the basis of a priori assumptions of their properties and also the properties of observables, remained on the margin and practically even outside the sphere of interest of specialists engaged in the creation of empirical models. This work is the result of attempts at constructing algorithmic procedures and programming systems for experiment designing and its data processing. Such systems cannot disregard generally utilized requirements involving mathematical modelling in physics and technology, i.e., invariance concerning a defined rotation, translation or gauge groups and also the tensor homogeneity of a model.

These problems have been investigated independently by dimensional analysis [19] and the theory of invariants [61]. In utilized mathematical procedures this would require the construction of algorithms in various languages (various spaces). This inconvenience has been removed by the construction of dimensional spaces, based on the theory of fiber bundles, proposed in this work.

As a result we have presented consequently elaborated algorithmic procedures to service empirical works intended to construct mathematical models both for cases investigated in accordance with the traditional dimensional analysis, the planning of experiments and the processing of obtained results in which scalar variables are used, and – so far unconsidered models with variables comprising tensors of a higher order than zero (Chapters 1, 2, 3, 4, 5 and 6).

Problems concerning the similarity and designing of models, as well as the construction of falsification procedure of hypotheses relating to the completeness of lists of variables describing a process, have been examined in both cases.

Chapter 6 occupies an intermediate position. It includes a solution of the old problem of selection of dimensional bases present in every formulation of the classical theorem-$\pi$. However, the proposed new metod (the universal graph) is constructed entirely with the help of dimensional geometry related to the classical idea of Drobot spaces. Many relatively simple examples depict the way the universal graph works.

With the appearance of fractals an important class of mathematical tools has become widely applied to modelling techniques. The second part of the book (chapters 7, 8 and 9) is devoted to the special version of dimensional analysis oriented onto models employing fractals and scaling dependences. The particular, new model of fractal measurement has been elaborated as well as the new projective dimensional analysis. Due to specific form of equations, nonexisting in classical dimensional analysis, some new techniques are proposed. Finally, Chapter 9 comprises exemplary applications of described methods. Concordant with the interests of authors examples concern mainly properties of materials as well as the structure and evolution of biological systems.

Some sections of this book (Chapter 1 and parts of Chapter 2, 4 and 10) were published by World Scientific in 1990 under the tittle *Dimensional Analysis in the Identification of Mathematical Models*. We express our thanks to the World Scientific for the permission to use excerpts from the above mentioned edition.

We express our sincere thanks to Mr Jan Rudzki for the English translation of this book.

Authors

## Contents

# CHAPTER 1

# DROBOT'S DIMENSIONAL SPACE AND A CLASSICAL THEORY OF MEASUREMENT

The construction of empirical mathematical models necessitates the introduction of mathematical operations on results of observations. This inclines us to treat measured quantities as elements of a set of a mathematical space. Both the theory of systems and the identification theory treat observation results as real numbers. This approach to measurement results has been established by the classical theory of measurement although it practically solved merely the problem of the construction of measurement scales, whereas the determination of the additivity of measured quantities already caused some trouble. We shall introduce the concept of dimensional space according to S. Drobot [19] because it defines, in a sufficiently explicit fashion, operations on measurement results or – more accurately – on the set of dimensional quantities interpreted as, for instance, physical magnitudes.

## 1.1. DROBOT'S DIMENSIONAL SPACE [19]

To construct a consequent algebraic scheme of dimensional analysis Drobot introduces the notion of dimensional space. It included such elements as dimensional quantities interpreted as quantities operated by quantitative theories of processes.

**Definition 1.1.** *The algebraic system* $\Pi = \left(\hat{\Pi}, \odot, \left\{^{[a]}\right\}_{a \in R}\right)$ *will be called a dimensional space if the following conditions are satisfied:*

1. $(\hat{\Pi}, \odot)$ *is an abelian group,*
2. *The powers* $\left\{^{[a]}\right\}$ *fulfill axioms of multiplication by a scalar for vector spaces (written in the form of powers):*

$$\forall a, b \in R \;\; \forall \hat{X}, \hat{Y} \in \hat{\Pi} \qquad \begin{aligned} \hat{X}^{[a+b]} &= \hat{X}^{[a]} \odot \hat{X}^{[b]}, \\ (\hat{X} \odot \hat{Y})^{[a]} &= \hat{X}^{[a]} \odot \hat{Y}^{[a]}, \\ (\hat{X}^{[a]})^{[b]} &= \hat{X}^{[ab]}, \\ \hat{X}^{[1]} &= \hat{X}, \end{aligned}$$

*3.* $\hat{\Pi}_0 = R_+ \in \hat{\Pi}$

*4. In the subset* $\hat{\Pi}_0 = R_+$ *of space* $\hat{\Pi}$ *the multiplication is identical with the multiplication of real numbers* ($\odot / R = " \bullet "$)

*5. The powers* $\{^{[a]}\}_{a \in R}$ *of elements of the subset* $\hat{\Pi}_0$ *are ordinary powers of the numbers* ($\hat{X}^{[a]} / R_+ = X^a$).

Henceforth, to simplify notation, we shall replace $\odot, \{^{[a]}\}$ with $\bullet, a$ also in the set of elements that are not elements of the subset $\hat{\Pi}_0$.

**Definition 1.2.** *The elements* $\hat{Z}_1, \hat{Z}_2, \dots, \hat{Z}_m \in \hat{\Pi}$ *are called dimensionally dependent if the numbers* $a_1, a_2, \dots, a_m \in R$ *not all equal to zero exist and are such that*

$$\prod_{i=1}^{m} \hat{Z}_i^{a_i} = \alpha, \quad \alpha \in \hat{\Pi}_0 \tag{1.1}$$

*If this condition does not occur (when (1.1) is fulfilled only for* $a_1 = a_2 = \dots = a_m = 0$ *and* $\alpha = 1$*) we say that the elements* $\hat{Z}_1, \hat{Z}_2, \dots, \hat{Z}_m$ *are dimensionally independent.*

**Definition 1.3.** *We say that the dimensional space* $\Pi$ *contains* $n$ *units (denoted by the symbol* $\Pi_n$*) if there exist* $n$ *dimensionally independent elements in it and every* $n+1$ *system of elements is dimensionally dependent. We shall also describe every system of dimensionally independent* $n$ *elements in the dimensional space* $\Pi_n$ *as the base of this space.*

Let $\hat{E}_1, \hat{E}_2, \dots, \hat{E}_n \in \hat{\Pi}_n$ be the system of units of dimensional space $\Pi_n$. The element $\hat{Z} \in \hat{\Pi}_n$ may then be presented consistent with definitions 1.2 and 1.3, in the form:

$$\hat{Z} = Z \prod_{k=1}^{n} \hat{E}_k^{z_k}; \quad Z \in \hat{\Pi}_0, \quad z_k \in R. \tag{1.2}$$

Dimensional space is not included in most works dealing with dimensional analysis [11], [60]. Measurement is treated as explicit mapping on $R$; if $\hat{\Pi}$ is the set of quantities of interest to us, the mapping referred to $f : \hat{\Pi} \longrightarrow R_+$.

It is not easy to decide, on the basis of Definition 1.2, whether the system of quantities $\hat{Z}_1, \hat{Z}_2, \dots, \hat{Z}_m \in \hat{\Pi}_n$ is dimensionally dependent or independent. We shall present a method of solving this problem for the dimensional space $\Pi_n$ spanned by the system of units $\hat{E}_1, \hat{E}_2, \dots, \hat{E}_n$.

**Theorem 1.1.** *Dimensional quantities* $\hat{Z}_1, \hat{Z}_2, \dots, \hat{Z}_m$ *expressed in the system of units* $\hat{E}_1, \hat{E}_2, \dots, \hat{E}_n \in \hat{\Pi}_n$ *by formulas*

$$\hat{Z}_i = Z_i \prod_{k=1}^{n} \hat{E}_k^{z_{ik}}, \quad Z_i \in \hat{\Pi}_0, \ \ z_{ik} \in R, \ \ i = 1, \ 2, \dots, m; \tag{1.3}$$

*can be treated as dimensionally independent if and only if the matrix made up from exponents $z_{ik}$, $i = 1, 2, \ldots, m$, $k = 1, 2, \ldots, n$ appearing in formulas (1.3) is of the $m$ order.*

Proof of this theorem may be found in work [19] (it is also known in linear algebra).

In the classical theory of measurement the possibility of measurements by different observers has been taken under consideration. It was stipulated that both descriptions of this observed system were to be univocal (cf. for instance, the "theory of uniqueness" cited by Berka [8] after Morgenstern and von Neuman [67]); in a dimensional analysis this corresponds to a situation in which two observers use two different systems of units (bases) of the same dimensional space. Let us examine an example when two systems of units are known, namely:

$$\hat{E}_1, \hat{E}_2, \ldots, \hat{E}_n \quad \text{and} \quad {}'\hat{E}_1, {}'\hat{E}_2, \ldots, {}'\hat{E}_n \quad \in \hat{\Pi}_n.$$

Using the formulas (1.3) we can express the units $\hat{E}_i$, $i = 1, 2, \ldots, n$, univocally in the system of units ${}'\hat{E}_k$, $k = 1, 2, \ldots, n$ and vice versa. We shall have

$$\hat{E}_i = E_i \prod_{k=1}^{n} {}'\hat{E}_k^{e_{ik}}, \quad e_{ik} \in R; \ \ E_i \in \hat{\Pi}_0, \ \ i = 1,\ 2, \ldots, n; \tag{1.4}$$

and

$${}'\hat{E}_k = {}'E_k \prod_{i=1}^{n} \hat{E}_i^{e'_{ik}} \quad e'_{ik} \in R; \ \ {}'E_k \in \hat{\Pi}_0 \ \ k = 1,\ 2, \ldots, n. \tag{1.5}$$

The matrices of exponents $\{e_{ik}\}$ and $\{e'_{ik}\}$ are of the $n$ order, namely $det\{e_{ik}\} \neq 0$ and $det\{e'_{ik}\} \neq 0$. Every element of this space can be expressed univocally utilizing formula (1.2) both in base $\hat{E}_i$ and ${}'\hat{E}_k$, that is

$$\hat{Z} = Z \prod_{i=1}^{n} \hat{E}_i^{z_i}, \quad Z \in \hat{\Pi}_0, \ \ z_i \in R, \tag{1.6}$$

$$\hat{Z} = Z' \prod_{k=1}^{n} {}'\hat{E}_k^{z'_k}, \quad Z' \in \hat{\Pi}_0, \ \ z'_k \in R. \tag{1.7}$$

If we know the dimensional quantity $\hat{Z} \in \hat{\Pi}_n$ in the base $\hat{E}_i \in \hat{\Pi}_n$ and we know its numerical measure $Z$ and exponents $z_i$ in relation to base $\hat{E}_1, \hat{E}_2, \ldots, \hat{E}_n$ we may – at the transformation of bases assigned by formula (1.4) – find $Z'$ and the exponents $z'_k$ of the dimensional quantity $\hat{Z}$ in a new system of units ${}'\hat{E}_1, {}'\hat{E}_2, \ldots, {}'\hat{E}_n$, namely:

$$\hat{Z} = Z \prod_{i=1}^{n} \left( E_i \prod_{k=1}^{n} {}'\hat{E}_k^{e_{ik}} \right)^{z_i} = Z \prod_{i=1}^{n} E_i^{z_i} \prod_{k=1}^{n} {}'\hat{E}_k^{\sum_{i=1}^{n} e_{ik} z_i}. \tag{1.8}$$

Comparing formulas (1.8) with (1.7) it is obvious that

$$Z' = Z \prod_{i=1}^{n} E_i^{z_i}, \quad z'_k = \sum_{i=1}^{n} e_{ik} z_i. \tag{1.9}$$

The formulas (1.9) provide, therefore, a method of transformation in the description of magnitude $\hat{Z}$ from one system of units into another.

In his work [19] Drobot defined the so-called dimensional transformation which in space $\Pi_n$ spanned over a fixed system of units $\hat{E}_1, \hat{E}_2, \ldots, \hat{E}_n$ assign to the element $\hat{Z} \in \hat{\Pi}_n$ a unique different element $\hat{Z}' \in \hat{\Pi}_n$.

**Definition 1.4.** *The transformation* $\hat{Z}' = \Theta(\hat{Z})$ $(\Theta : \hat{\Pi}_n \longrightarrow \hat{\Pi}_n)$ *such that for* $\hat{Z} = Z \prod_{i=1}^{n} \hat{E}_i^{z_i}$ *and* $\hat{Z}' = Z' \prod_{i=1}^{n}{}' \hat{E}_i^{z'_i} \in \hat{\Pi}_n$, $Z'$, $z'_i$ *are linked with* $Z$ *and* $z_i$ *via dependencies (1.9) will be called a dimensional transformation.*

The dimensional transformation satisfies the following conditions:

i) $\Theta(\hat{Z}_1, \hat{Z}_2) = \Theta(\hat{Z}_1)\Theta(\hat{Z}_2); \quad \hat{Z}_1, \ \hat{Z}_2 \in \hat{\Pi}_n,$

ii) $\Theta(\hat{Z}^a) = \Big\{\Theta(\hat{Z})\Big\}^a; \quad \hat{Z} \in \hat{\Pi}_n, \ a \in R,$

iii) is one to one

iv) $\Theta(\alpha) = \alpha, \quad \alpha \in \hat{\Pi}_0.$

**Definition 1.5.** *The two quantities* $\hat{X}$, $\hat{Y} \in \hat{\Pi}$ *have the same dimension if* $\hat{X}\hat{Y}^{-1} \in \hat{\Pi}_0$ *i.e., it is a dimensionless quantity.*

Using symbols introduced by Maxwell, we shall denote this fact by writing

$$[\hat{X}] = [\hat{Y}] \subset \hat{\Pi}.$$

The relation described in Definition 1.4 divides elements of dimensional space into disconnected classes, so that the same classes encompasses only elements of the same dimensions, e.g. $[5\ m] = [3\ km] = [2\ inches] = \ldots$.

The algebraic structure of space $\Pi$ is such that group $R_+$ is a normal subgroup of $(\hat{\Pi}, \odot)$, consequently $\Pi$ as group is isomorphic with the product:

$$\hat{\Pi} = \hat{\Pi}/R_+ \times R_+.$$

Summation may be used in dimensional space only for elements of the same dimension. Thus:

$$\alpha\hat{X} \pm \beta\hat{X} = (\alpha \pm \beta)\hat{X} \tag{1.10}$$

where $\alpha, \beta \in R_+$, $\hat{X} \in \hat{\Pi}$ represent elements of the examined dimensional class. However, consistent with Definition 1.1 in the case of the real number $\gamma < 0$ the product $\gamma \bullet \hat{X} \notin \hat{\Pi}$. Intuitively, the construction of space would be clear if the whole $R$ space were to be introduced into space $\hat{\Pi}$ as a subset, we would have to interpret, e.g., the quantity $(-3m)^{1/2}$, but in fact this quantity is also not an element of the set $\hat{\Pi}$. This is an essential drawback of the space here discussed from the aspect of use, for instance, in physics (where we assume rotations and translations of co-ordinate systems).

The limit of a sequence $\left\{\alpha_n \hat{X}\right\}$, where $\{\alpha_n\}$ is the numerical sequence, is perceived just like the sum (1.10), namely:

$$\lim_{n\to\infty}\left(\alpha_n \hat{X}\right) = \left(\lim_{n\to\infty}\alpha_n\right)\hat{X}. \tag{1.11}$$

It should also be noted that (1.10), (1.11) and Definition 1.5 show that

$$\begin{aligned} \left[\tfrac{d\hat{Y}}{d\hat{X}}\right] &= \left[\hat{Y}\hat{X}^{-1}\right], \\ \left[\int \hat{Y} d\hat{X}\right] &= \left[\hat{Y}\hat{X}\right]. \end{aligned} \tag{1.12}$$

The class of dimensionally homogeneous and invariant functions (see Definition **??**) is closed under differentiation but open with respect to integration. Therefore (1.12) holds if the both sides of equations are properly defined.

## 1.2. PROBLEMS OF THE CLASSICAL THEORY OF MEASUREMENT

The classical theory of measurement (cf. Campbell [3], [8]) refers to a mapping of the empirical relational system into a numerical relational system. Speaking of an empirical relational system we have in mind a certain empirical set, the relation determining the linear order and the empirical operation $\oplus$ or $\odot$ on the elements of this set. By a numerical system we understand, e.g., $\{R_+, \leq, +\}$, $\{R_+, \leq, \bullet\}$. We introduce here the condition of correspondence between the empirical and the numerical system based on definitions of a similar order quoted here after Kuratowski [39].

**Definition 1.6.** *We say that the relation $\leq$ subordinating set $A$ and the relation $\leq^*$ subordinating set $A^*$ determine similar arrangements (or that $A$ and $A^*$ are isomorphic) if there exists a one-to-one mapping $f$ of the set $A$ into $A^*$ fulfilling the equivalence*

$$(x \leq y) = (f(x) \leq^* f(y)),$$

*that is, if the element $x$ precedes the element $y$ in set $A$ then, and only then, when the element $f(x)$ precedes $f(y)$ in set $A^*$. The mapping $f$ is, therefore, simultaneously defined as $f : A \longrightarrow A^*$.*

It can easily be proven that the similarity relation introduced by Definition 1.6 is an equivalent relation. If the mapping $f$ is one-to-one and transforms the empirical set into a numerical set then (it was done likewise by Rosen [51] on the basis of a well founded notion of length measurement) we shall call it a *"meter"*. In fact, the principal problem of the classical theory of measurement involves a logical reconstruction of the measurement operation and an examination of properties of measurement scales used for this purpose. But attempts have also been made at introducing on this basis a division of measured quantities into extensive (additive) and intensive (nonadditive) quantities.

If an empirical set has to be put in order we shall introduce a binary relation. This introduces (according to Rosen [51]) a one-to one transformation referred to in Definition 1.6. Such a transformation assigns some numerical values to a certain individual characteristic of elements of the investigated empirical set. The order relation is introduced for that reason, using the language of dimensional space, on elements belonging to the same dimensional class ( cf. Definition 1.5). Consequently, the measurement theory can solve the problem: will two objects measured with a meter, one being, for instance, two meters and the other three meters long, measure five meters when linked together (if the measurement scale is properly constructed) ? But the problem whether the operation of composition is sensible in an explicit empirical situation, remains unsolved. For that reason, we must differentiate two separate aspects of the here discussed notion of additivity, namely, the construction of the measurement scale which ensures an identical measurement result of the two composed "objects" with the sum of measurement results made separately on these two "objects" – differentiated from the sensibleness of the operation of putting objects together. The last problem cannot be solved on the basis of measurement theories, we may assume that it is insoluble on the basis of any formal theory involving measurement.

As we know, the additivity problem involving the construction of measurement scales has been solved in the classical measurement theory [8] by the introduction of so-called linear interval scales and ratio scales. They were somehow differently interpreted by von Neuman and Morgenstern [67] who did not introduce compositions into the empirical system. Two mappings were considered:

$$\begin{array}{llllll} f: u & \longrightarrow & \rho, & u \in U, & \rho \in R_+, & \rho = f(u), \\ f': u & \longrightarrow & \rho', & u \in U, & \rho' \in R_+, & \rho' = f'(u). \end{array} \tag{1.13}$$

To fulfill the postulate of uniqueness the mapping (1.13) must be such that if $u > w, \;\; u, w \in U$ then:

$$f(u) > f(w) \quad \text{and} \quad f'(u) > f'(w). \tag{1.14}$$

We must also describe the transformation:

$\rho \rightleftharpoons \rho'$ which is one to one and can be written in the form:

$$\rho' = \phi(\rho). \tag{1.15}$$

Let us take the pair $u, w \in U$ while $u > w$ (each pair in the set $U$ satisfies one of the relations $u > w$, $u < w$ or $u = w$) and the mapping $f$, $f(u) = \rho$, $f(w) = \sigma$, then obviously $\rho > \sigma$. Let us introduce on the numbers $\rho$ and $\sigma$ the operation:

$$\alpha\rho + (1-\alpha)\sigma \tag{1.16}$$

in which $\alpha \in R_+$ is from the interval $(0 < \alpha < 1)$.

To maintain condition (1.14) and operation (1.16) the transformation $\phi$ should keep the following properties:

$$\begin{gathered} \phi(\rho) > \phi(\sigma), \\ \phi\{\alpha\rho + (1-\alpha)\sigma\} = \alpha\phi(\rho) + (1-\alpha)\phi(\sigma). \end{gathered} \tag{1.17}$$

It has been proved (in [67] with certain assumptions unessential for our considerations) that the transformation $\phi$ (1.15) fulfilling the properties (1.17) is a linear transformation, i.e.,

$$\rho' = \phi(\rho) = a_0\rho + a_1. \tag{1.18}$$

It should be noted that the transformation $\phi$, just like the transformation $\Theta$ (cf. Definition 1.4) fulfills the condition iv) $\Theta(\alpha) = \alpha$, $\phi(\alpha) = \alpha$, $\alpha \in \hat{\Pi}_0$. The numbers $\rho$ and $\sigma$ are the numerical measures. As regards $a_1 \neq 0$ we have to do with so-called interval scales and, as regards $a_1 = 0$ with ratio scales.

If we treat the measurement result (with the aid of a ratio scale) as an element of space $\Pi$, denote the measurement results (1.13) respectively $\hat{\rho}, \hat{\sigma}$, and produce their quotient, i.e.,

$$\frac{\hat{\rho}}{\hat{\sigma}} = \beta,$$

it becomes evident that on the strength of Definition 1.5 $\beta \in \hat{\Pi}_0$ and, of course

$$\frac{\hat{\rho}}{\hat{\sigma}} = \frac{\hat{\rho}'}{\hat{\sigma}'} = \beta. \tag{1.19}$$

The property (1.19) will be maintained at measurements involving the linear-interval scale when $\hat{\rho}$ and $\hat{\sigma}$ denote measurement results of increments of a certain quantity. As regards quantitative theories of physics views have often been expressed not to use manmade scales of descriptions. We can choose from set $U$ (corresponding in the dimensional space $\Pi$ to the class of elements of the same dimension) a certain element (standard) and treat it as a standard "meter", measurement results will then be direct equivalents of the number $\beta$. This well known idea has been used in dimensional analyses for the construction of quantitative models of processes.

Returning to the subject, it should be noted that von Neuman and Morgenstern achieved "additivity" in the sense of scale property without postulating the composition operation (concatenation in the theory of measurement) in the empirical set (operation (1.16) has been introduced for the set of mappings). Nonetheless, we have found that the additivity problem, taken broadly, cannot be solved in the measurement theory and the question arises how to solve it.

Quantities measured in formalized (e.g., physical) theories are treated as dimensional quantities (space $\Pi$) and, at the same time, as tensors of a corresponding valence. Consequently, the additivity problem should be examined also formally not only on the ground of a corresponding mathematical model but also on the basis of a certain interpretation context of a given theory.

Let us formulate, after work [30] three postulates describing the additivity of an examined quantity which should be fulfilled simultaneously, namely:

1. The so-called dimensional quantity which is an element of space $\Pi$ is the mathematical model of measured magnitudes. Formally we may allow there an addition and substraction operation on quantities of the same dimension (cf. Formula (1.10)). Is this a sufficient condition ? If this were so then, for example, a work defined as the scalar product of force $\times$ path and torque defined as the vector product of a force $\times$ arm would be treated as additive quantities. As we know, these two quantities have the same dimension, i.e., $\hat{N} \cdot \hat{m}$ (Newton meter).
2. In respective mathematical models each quantity is a tensor of a specific valence. In physics there is an obligatory condition of the so-called tensor homogeneity. It requires each component of the sum to be a tensor of the same valence. It prohibits an addition of the scalar and the tensor (i.e., work and moment). It should be noted that both the theory of measurement and the model of dimensional space, according to [19], deal only with scalar quantities.
3. The examined quantity may be recognized as additive when it satisfies condition 1 and 2, moreover, when the interpretation contents of the theory indicates that the addition and substraction operation is sensible.

To elucidate this problem let us quote an example used by both Campbell [13] and Berka [8] involving the connection of resistances in direct current circuits. There is the resistance $\hat{R} = \hat{R}_1 + \hat{R}_2$, in a series connection, and $\hat{R} = \frac{\hat{R}_1 \hat{R}_2}{\hat{R}_1 + \hat{R}_2}$ for a parallel connection. If resistances are linked then postulates 1 and 2 are fulfilled. However, the formulas:

$$\hat{R} = \hat{R}_1 + \hat{R}_2,$$
$$\hat{R} = \frac{\hat{R}_1 \hat{R}_2}{\hat{R}_1 + \hat{R}_2}, \tag{1.20}$$

have been obtained after required transformations from a summing of the voltage in the first case and of currents – in the second case. The sensibleness of this summing results from interpretation postulates of the relevant theory and is compatible with postulate 3. It is easy to see that formal operation of addition occurs in both formulas (1.20). In calculations its admissibility is decided by postulates 1 and 2. Later we shall discuss the possibility of reducing postulates 1 and 2 to a single one after the enhancement of the dimensional space.

According to Berka the numerical system, e.g., $\{R_+, \leq, +\}$ is indispensable for a formal description of the system. This view may or may not be correct. In reality, the "numerical system" thus understood describes measurement results in one specific coset and even if the operation $\oplus$ or $\odot$ were sensible there, it would not solve the possibility of introducing these operations to the construction of equations describing the state of a system because many cosets or (to use Berka's language) "numerical systems" must be used to describe this system.

The measurement theory does not define the relation between "objects" of various cosets (dimensional classes) and operations (e.g., composition) on various objects. These difficulties will not appear if tasks of the measurement theory are reduced to a correct construction of measurement scales, nor is it necessary to examine different specific empirical sets – this view has been expressed by von Neuman and Morgenstern [67]. The necessity to introduce operations on elements of various cosets occurs only when an attempt is made at creating a formal description of a system's behavior where the correspondence with the "empirical system" is much more complicated than the one referred to in the theory of measurement. In the description of systems many "empirical objects"are defined by constitutive rules (rules of correspondence in the methodology of science specifying theoretical notions), referring not only to empirical actions but also to mathematics, this concerns the notion of force, moment, energy, etc. The classical theory of measurement also speaks of a division of measurements into so-called fundamental and derivative measurements and, consequently, into fundamental and derivative quantities. This has led to essential complications involving definitions indicated, among others, by Popper [47], who referred to a vicious circle of operative definitions. We sense intuitively that quantities used to define all remaining (magnitudes, terms, notions) appearing in the descriptions of a process (system) relate to the notion of basic quantities. In formal models, for instance in Drobot's dimensional space, there appears an explicit term "base"of dimensional space. Just as in linear spaces and in Drobot's space a base can be usually distinguished in many ways. Practically, this also concerns physical theories unless a certain agreement supported by a rule ( e.g., the S.I. system) limiting the selection of a base is accepted.

In dimensional space we may select out of $s$ elements $\hat{Z}_l \in \hat{\Pi}_n$, $l = 1, 2, \ldots, s$, $s \geq n$ an $m$ element base $m \leq n$, usually in many ways. Let us assume that the elements $\hat{Z}_1, \hat{Z}_2, \ldots, \hat{Z}_m$ create a base (cf. Definition 1.2) and we cannot select an $m + 1$ element base out of elements $\hat{Z}_l$, $l = 1, 2, \ldots, s$. Then, the remaining quantities $\hat{Z}_l$ on the strength on Definition 1.2 and 1.3:

$$\hat{Z}_{m+j} = \phi_j \prod_{i=1}^{m} \hat{Z}_i^{a_{ji}}, \quad a_{ji} \in R, \quad \phi_j \in \hat{\Pi}_0, \quad j = 1, 2, \ldots, s - m. \tag{1.21}$$

Consequently, the quantities $\hat{Z}_i$ fulfill in formulas (1.21) the function of basic magnitudes which help to "define" the derivative magnitude $\hat{Z}_{m+j}$.

In the description of systems we endeavor to attain a mathematical model invariant as regards measurement scales invented by man. This indicates attempts to measure variables describing the state of a system by some kind of internal measure – characteristic of the system (see the discussion concerning this problem, e.g., in work [72]). This experience is closely connected with the solving of partial differential equations. In astrophysics, for instance, scientists often introduce a certain individuated mass, length and time (e.g., the mass of a planet, etc.), changing (what is easy to show) original equations in this way.

Let us assume that we describe a system the state of which is univocally specified by the variables $\hat{Z}$, $\hat{Z}_1, \hat{Z}_2, \ldots, \hat{Z}_s \in \hat{\Pi}_n$. The equivalence relation introduced by Definition 1.5 produces in set $\hat{\Pi}_n$ a division into disconnected subsets $\hat{\Pi}_n/R_+$. Let us suppose that in the subset of a $[\hat{Z}_{p+1}]$ dimension there exist at least two elements: $\hat{Z}_{p+1}$ and $\hat{Z}_p$ ($[\hat{Z}_{p+1}] = [\hat{Z}_p]$). The element $\hat{Z}_{p+1}$ can be treated as a measurement unit "meter", then:

$$\hat{Z}_p = \alpha \hat{Z}_{p+1}. \tag{1.22}$$

It is easy to show that $\alpha \in \hat{\Pi}_0$. The result of the measurement by the $\hat{Z}_{p+1}$ "meter" will be described as $f(\hat{Z}_p) = \alpha$ consistent with (1.22)

$$\alpha = \frac{\hat{Z}_p}{\hat{Z}_{p+1}} \tag{1.23}$$

because $[\hat{Z}_{p+1}] = [\hat{Z}_p]$, so $\alpha \in \hat{\Pi}_0$.

However, both the "meter" $\hat{Z}_{p+1}$ and the quantity $\hat{Z}_p$ can be written in the system of units $\hat{E}_k$ and $'\hat{E}_i \in \hat{\Pi}_n$ (cf. formulas (1.7) and (1.8)).

Having inserted into (1.22) $\hat{Z}_p$ and $\hat{Z}_{p+1}$ recorded in the system of units $\hat{E}_k$ we get $\alpha = \hat{Z}_p/\hat{Z}_{p+1}$; when $\hat{Z}_p$ and $\hat{Z}_{p+1}$ are written in turn in the $'\hat{E}_i$ system and the formula (1.8) is taken into account we again obtain $\alpha = \hat{Z}'_p/\hat{Z}'_{p+1}$ we have shown here, by the way, that the previously recorded formula (1.19) is correct. But what should be done if only one element appears in the class of the $\hat{Z}_p$ dimension? There are two possibilities compatible with the two concepts of the "meter" constructions, i.e.:

1. The construction of a "meter" on a base selected among quantities describing the state of a system, that is the quantities $\hat{Z}$, $\hat{Z}_1, \hat{Z}_2, \ldots, \hat{Z}_s$.
2. The adding of the quantity $\hat{Z}_{s+1}$ ($[\hat{Z}_{s+1}] = [\hat{Z}_p]$) to the set $\hat{Z}$, $\hat{Z}_1, \hat{Z}_2, \ldots, \hat{Z}_s$ i.e., the "meter" for the quantity $\hat{Z}_p$. This is essential for a mathematical description, a new variable appears in it, the mathematical model will differ therefore, from the model in case 1. If there are no essential physical reasons to distinguish variable $\hat{Z}_{s+1}$ in the description of the state of a system, the possibility of using this procedure must be rejected.

It suffices to construct a "meter" in accordance with formula (1.21) for magnitude $\hat{Z}_p$. It is invariant in relation to possible measurement unit systems because a "meter":

$$\hat{Z}_{s+1} = 1 \prod_{i=1}^{m} \hat{Z}_i^{a_{s+1,i}} \tag{1.24}$$

and $\hat{Z}_p = \alpha \hat{Z}_{s+1}$ ($f(\hat{Z}_p) = \alpha$), and more explicit:

$$\alpha = \frac{\hat{Z}_p}{\prod_{i=1}^{m} \hat{Z}_i^{a_{s+1,i}}}, \quad \alpha \in \hat{\Pi}_0. \tag{1.25}$$

It is easy to check that $\alpha$ does not depend on the accepted system of units. In this case, the $\alpha$ value can depend only on the selection of the base (this problem will be resumed in Chapter **??**).

We wish to indicate, moreover, that both the change of the system of units in the description of dimensional quantities (transformation $\Theta$) and a change of the base (in the formal sense these two procedures are identical) satisfy condition (1.18) imposed on the transformation (1.15) in the theory of measurements. Let us measure, therefore, the dimensional quantity $\hat{Z}$ with a "meter"$= 1 \prod_{k=1}^{n} \hat{E}_k^{z_k}$, we shall obtain $f(\hat{Z}) = Z$ (cf formula (1.6)) and with an "inch"$= 1 \prod_{i=1}^{n} {}'\hat{E}_i^{z_i}$, where $\left[1 \prod_{k=1}^{n} \hat{E}_k^{z_k}\right] = \left[1 \prod_{i=1}^{n} {}'\hat{E}_i^{z_i}\right]$ we shall have $\overset{*}{f}(\hat{Z}) = Z'$. We know that $Z$ and $Z'$ are linked with the dependence (1.9) (cf. Definition 1.4). The transformation (1.15) should guarantee that

$$Z' = \phi(Z) = a_0 Z \tag{1.26}$$

for the ratio scale (and also for interval scales if we interpret $Z$ as an increment). Calculating from (1.26) $a_0$ and considering (1.9) we get:

$$a_0 = \frac{Z'}{Z} = \prod_{i=1}^{n} E_i^{z_i} \tag{1.27}$$

In this case, the constant $a_0$ of the linear transformation depends nonlinearly on the rescaling of the system of units, but it is constant and uniform for each of the "new meter"$= \prod_{i=1}^{n} E_i^{z_i} =$"inch".